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Elliptic Littlewood identities

Eric M. Rains
Department of Mathematics, California Institute of Technology

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Abstract

We prove analogues for elliptic interpolation functions of Macdonald's version of the Littlewood identity for (skew) Macdonald polynomials, in the process developing an interpretation of general elliptic "hypergeometric" sums as skew interpolation functions. One such analogue has an interpretation as a "vanishing integral", generalizing a result of [17]; the structure of this analogue gives sufficient insight to enable us to conjecture elliptic versions of most of the other vanishing integrals of [17] as well. We are thus led to formulate ten conjectures, each of which can be viewed as a multivariate quadratic transformation, and can be proved in a number of special cases.

Contents

1 Introduction

In recent work of the author [14] (see also [5] for an independent treatment), a family of "interpolation functions" were introduced, generalizing Okounkov's interpolation polynomials [11], which in turn generalize shifted Macdonald polynomials [20] and Macdonald polynomials [10] themselves. Among the identities satisfied by the interpolation functions is an analogue of the Cauchy identity, which for Macdonald polynomials states

$$\sum_\mu P_\mu(x_1,\dots,x_n;q,t)P_{\mu'}(y_1,\dots,y_m;t,q) = \prod_{\substack{1\le i\le n\\ 1\le j\le m}} (1+x_iy_j). \tag{1.1}$$

Macdonald also proved (generalizing a result of Kadell for Jack polynomials) an analogue for Macdonald polynomials of the Littlewood identity, see [10, Ex. VI.7.4]:

$$\sum_\mu c_\mu(q,t) P_{\mu^2}(x_1,\dots,x_n;q,t) = \prod_{1\le i<j\le n} \frac{(tx_ix_j;q)}{(x_ix_j;q)}, \tag{1.2}$$

where P_λ is a Macdonald polynomial, μ^2 denotes the partition with parts $(\mu^2)_i = \mu_{\lceil i/2\rceil}$,

$$(x;q) := \prod_{k\ge 0}(1-q^k x), \tag{1.3}$$

and the coefficients $c_\mu(q,t)$ are given by an explicit product:

$$c_\mu(q,t) = \prod_{(i,j)\in\mu} \frac{1-q^{\mu_i-j}t^{2\mu'_j-2i+1}}{1-q^{\mu_i-j+1}t^{2\mu'_j-2i}}. \tag{1.4}$$

(The usual notation for $(x;q)$ would be $(x;q)_\infty$, but since we never use finite q-Pochhammer symbols, we suppress ∞ throughout.) This is the q,t-analogue of Littlewood's identity for Schur functions:

$$\sum_\mu s_{\mu^2}(x_1,\dots,x_n) = \prod_{1\le i<j\le n}(1-x_ix_j)^{-1}, \tag{1.5}$$

which describes the decomposition of $S^*(\wedge^2(\mathbb{C}^n))$ as a representation of GL_n, and thus by Frobenius reciprocity determines which irreducible representations of GL_{2n} have invariants under Sp_n (since the coordinate ring of the affine variety GL/Sp is obtained from $S^*(\wedge^2(\mathbb{C}^{2n}))$ by inverting the pfaffian). The purpose of the present note is to generalize such Littlewood-type identities to the elliptic level.

The primary obstacle to such an extension is the fact that, unlike the given form of the Cauchy identity, for which the terms vanish unless the partition μ is contained in an $m\times n$ rectangle, the Littlewood identity intrinsically involves a nonterminating sum. Unfortunately, at the elliptic level, infinite sums seem inevitably to encounter convergence difficulties, making a direct extension problematical. One must thus either modify the sum in such a way as to force termination (say by a suitable choice of the coefficients c_μ), or replace the sum by an integral. We will, in fact, take both approaches.

Our first step is to observe that Macdonald's Littlewood identity has a generalization (implicit in [10]; the argument sketched in Ex. I.5.27 and Ex. VI.7.6 op. cit. carries over mutatis mutandum) to skew Macdonald polynomials:

$$\sum_\mu c_\mu(q,t) P_{\mu^2/\lambda}(x_1,\dots,x_n;q,t) = \prod_{1\le i<j\le n}\frac{(tx_ix_j;q)}{(x_ix_j;q)} \sum_\mu c_\mu(q,t) Q_{\lambda/\mu^2}(x_1,\dots,x_n;q,t), \tag{1.6}$$

where $c_\mu(q,t)$ is as above. Of course, this in itself makes an extension more difficult, given the absence (but see below) of a good theory of skew versions of the interpolation functions. On the other hand, the proof of Macdonald's Littlewood identity uses only the case $n=1$ of this skew Littlewood identity, together with a corresponding case of the skew Cauchy identity. This case is particularly amenable to generalization, as both sums are finite (indeed, each has only one nonzero term), and the case $n=1$ of the skew Macdonald polynomials *does* have a very natural elliptic analogue. Indeed, the principal specialization (i.e., with variables specialized to $v,\dots,vt^{n-1}$) of a skew Macdonald polynomial can be expressed as a limit of an elliptic binomial coefficient, essentially just a value of an elliptic interpolation function. If one replaces the skew Macdonald polynomials by such elliptic binomial coefficients in the $n=1$ case, one finds that both sums still have only one surviving term, and one is led immediately to an elliptic analogue of the identity, Lemma 4.1 below.

To obtain a more general elliptic analogue, there are two natural approaches. The first is to develop a theory of skew interpolation functions, prove a corresponding skew Cauchy identity, then directly lift Macdonald's argument to the elliptic level. Roughly speaking, skew interpolation functions should give the coefficients in a generalized branching rule:

$$\begin{aligned}\mathcal{R}^{*(n+m)}_{\boldsymbol{\lambda}}&(x_1,\dots,x_n,y_1,\dots,y_m;t_0,u_0;t;p,q)\\&=\sum_{\boldsymbol{\mu}}\mathcal{R}^{*(m,n)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(y_1,\dots,y_m;t_0,u_0;t;p,q)\,\mathcal{R}^{*(n)}_{\boldsymbol{\mu}}(x_1,\dots,x_n;t_0,u_0;t;p,q).\end{aligned}\tag{1.7}$$

(In contrast to the Macdonald case, these coefficients depend on n in a slightly nontrivial way. Also, recall from [15] that bold greek letters denote pairs of partitions.) Since these coefficients are understood for $m=1$ (a special case of [14, Thm. 4.16]), one could simply define skew interpolation functions by induction, giving an m-fold sum (in which each individual sum is over partitions). However, it turns out that one can use connection coefficients together with the existence of a special case of interpolation functions expressible as a product to obtain these coefficients via a single sum. Moreover, if the arguments $y_1,\dots,y_m$ contain partial geometric progressions of step t, the coefficients of the sum simplify accordingly, and one is thus led to the definition of Section 2. (See Theorem 2.5 for the relation between the skew interpolation functions so defined and ordinary interpolation functions; the remark following the theorem expresses the above expansion coefficients in terms of skew interpolation functions.) A suitable analytic continuation argument gives an analogue of the Cauchy identity (Theorem 3.7), and then Macdonald's argument lifts to give an elliptic Littlewood identity, Theorem 4.4.

The other natural approach to an elliptic analogue is to retain the use of binomial coefficients (i.e., restrict one's attention to principally specialized skew Macdonald polynomials), but hope for an analogue with additional parameters. It turns out that enough degrees of freedom survive in the choice of coefficients that one can use those coefficients to enforce termination, giving Theorem 4.5 below. Moreover, the structure of the coefficients is such that one can analytically continue one of the two sums to a suitable contour integral, Theorem 4.7. This in turns suggests a further extension in which both sides are integrals, stated as Conjecture L1, for which we can prove a number of special cases.

The "integral=sum" version of the identity has a particularly striking interpretation coming from the fact that one can invert the elliptic binomial coefficients to move the sum inside the integral. The resulting sum of interpolation functions in the integrand then becomes a special case of the elliptic biorthogonal functions of [15, 14], and one thus deduces that a certain integral of such functions vanishes unless the indexing partition (or, rather, partition pair) has the form $\boldsymbol{\mu}^2$. This is the elliptic analogue of a result proved for Koornwinder polynomials in [17], and in fact gives a stronger result even at the Koornwinder level, since the techniques of [17] gave no information about the nonzero values. This suggests in turn that the other results of [17] involving the same vanishing condition should also be related to our elliptic Littlewood identity, and indeed we have been able to formulate two conjectures along those lines, Conjectures 1 and 2, which again hold in a number of special cases, and have three different results of [17] as limiting cases. In particular, every result of [17] that has $\lambda = \mu^2$ as the nonvanishing condition is a limit of either Corollary 4.9 or one of Conjectures 1 or 2. (We also give analogues for the results with condition $\lambda = 2\mu$, but it remains an open problem to lift the remaining vanishing theorems to the elliptic level, even conjecturally.) For instance, one limit of the latter conjecture is the fact that

$$\int P_\lambda(\dots, z_i^{\pm 1}, \dots; q, t) \prod_{1\le i<j\le n} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)}{(t z_i^{\pm 1} z_j^{\pm 1}; q)} \prod_{1\le i\le n} \frac{(z_i^{\pm 2}; q)}{(t z_i^{\pm 2}; q)} \frac{dz_i}{2\pi\sqrt{-1} z_i} \tag{1.8}$$

vanishes unless $\lambda = \mu^2$ for some μ, which in turn is a (q,t)-analogue of the representation-theoretic fact that the integral of a Schur function over the symplectic group similarly vanishes (equivalent by Frobenius reciprocity to the fact that only those Schur functions appear in the classical Littlewood identity).

Macdonald also gave a dual version of the Littlewood identity, in which rather than summing over partitions with even multiplicities, one sums over partitions with even parts. (Littlewood's original version of this identity gives the decomposition of $S^*(S^2(\mathbb{C}^n))$:

$$\sum_\mu s_{2\mu}(x_1, \dots, x_n) = \prod_{1\le i\le j\le n} (1 - x_i x_j)^{-1}, \tag{1.9}$$

and is related to the invariants of O_n inside irreducible representations of GL_n.) This dual Littlewood identity can, of course, be obtained from the usual Littlewood identity by simply applying Macdonald's involution to conjugate the partitions involved. One can naturally do the same for the elliptic Littlewood identities, but a new behavior arises. For the $\boldsymbol{\mu}^2$-type Littlewood identity, there is an analytical symmetry between the parameters p (specifying an elliptic curve) and q (specifying a point on that curve), which is broken by duality. If one attempts to restore this symmetry after dualizing, one finds that, in contrast to the $\boldsymbol{\mu}^2$-type Littlewood identity, which is a product of two equivalent identities, one p-elliptic, and one q-elliptic, the restoration of symmetry in the dual identity requires that one multiply by a conjectural q-elliptic identity which is *not* equivalent to the original dual identity. Moreover, this partner identity itself has a different broken symmetry, namely the natural action of $\mathrm{SL}_2(\mathbb{Z})$ as modular transformations of the family of elliptic curves. One thus finds that each of our identities and conjectures leads to a whole family of conjectures in this way; the Littlewood identity itself gives rise to three conjectural integral transforms, while the other vanishing conjectures correspond to seven different integral transforms. The latter group of conjectures (a single orbit under the various formal symmetries) is particularly interesting, as even without the interpolation functions in the integrands, they would give rise to new transformations of higher-order elliptic Selberg integrals (specifically, quadratic transformations). In particular, several of the special cases we prove give nontrivial identities of this form; of particular note is Theorem 5.10, which expresses certain 2- and 3-dimensional elliptic Selberg integrals as explicit linear combinations of univariate integrals and constants. See also [4], which proves the special case $\boldsymbol{\lambda} = 0$ of Conjectures Q3 and Q7 below. It can be shown (work in progress) that this implies the $\boldsymbol{\lambda} = 0$ cases of the remaining "Q" conjectures; for Conjecture Q1, this follows from Conjecture Q7 by Proposition 1.1 below, but the other cases require new machinery beyond the scope of the present work.

The plan of the paper is as follows. After a discussion of notation at the end of this introduction, we proceed in Section 2 to define our skew interpolation functions, and discuss a number of their properties, especially their connection to ordinary interpolation functions. (We also state a transformation of higher-order elliptic Selberg integrals (conjectural in the original version of this paper, since proved by Van de Bult) related to one of those properties, largely because the same conjecture arose in a different context while working on [15].) Then in Section 3, we discuss the corresponding analogues of the Cauchy identity, along with some necessary preliminaries concerning when skew interpolation functions can be guaranteed to vanish, thus making the relevant sums finite. Section 4 gives the two main forms of the elliptic Littlewood identity, as well as the three associated conjectures at the integral level. Finally, Section 5 discusses a number of conjectures related to the vanishing integrals of [17], with sketches of proofs of various special cases. Note that although this last section may seem on first glance to have drifted away from the theme of the paper, the corresponding "vanishing" conjectures, when degenerated to identities of Macdonald or Koornwinder polynomials, become Littlewood-type identities in a suitable limit as the number of variables tends to infinity. (More precisely, taking the limit $n \to \infty$ as in [13] gives either Macdonald's Littlewood identity, its dual, or an identity originally conjectured by Kawanaka [8] and recently proved in [9] (see also the discussion after Conjecture L3 below, which sketches an alternate proof).)

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Notation. We use the notation of [14] and [15]. In particular, bold-face greek letters refer to pairs of partitions; if only one of the partitions is nonzero, we will either give the partition pair explicitly, or rewrite using the notation of [14], explicitly breaking the symmetry between p and q. Thus, for instance, the interpolation functions are denoted by

$$\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(z_1,\dots,z_n;a,b;t;p,q), \tag{1.10}$$

which factors as

$$\mathcal{R}^{*(n)}_{\lambda,\mu}(z_1,\dots,z_n;a,b;t;p,q) = R^{*(n)}_{\lambda}(z_1,\dots,z_n;a,b;p,t;q)R^{*(n)}_{\mu}(z_1,\dots,z_n;a,b;q,t;p), \tag{1.11}$$

with the first factor q-elliptic, and the second p-elliptic. Relations and operations on single partitions extend to partition pairs in the obvious way; in particular, $\boldsymbol{\lambda} \subset \boldsymbol{\mu}$ denotes the product of the usual inclusion orders on the two pieces. We will need some additional notations for partitions. Of particular importance are λ^2, denoting the partition with $\lambda^2_i = \lambda_{\lceil i/2\rceil}$, and 2λ, denoting the partition with $(2\lambda)_i = 2\lambda_i$, both extending immediately to partition pairs. If $\lambda_1 \le m$, then $m^n \cdot \lambda$ denotes the partition with

$$(m^n \cdot \lambda)_i = \begin{cases} m & i \le n \\ \lambda_{i-n} & i > n. \end{cases} \tag{1.12}$$

If $\ell(\lambda) \leq n$, then $m^n + \lambda$ denotes the partition with

$$(m^n + \lambda)_i = m + \lambda_i. \tag{1.13}$$

Finally, if $\lambda_1 \leq m$ and $\ell(\lambda) \leq n$, then

$$(m^n - \lambda)_i = m - \lambda_{n+i-1}. \tag{1.14}$$

We specifically recall the elliptic Gamma function

$$\Gamma_{p,q}(z) := \prod_{0 \leq i,j} \frac{1 - p^{i+1} q^{j+1}/z}{1 - p^i q^j z}, \tag{1.15}$$

with the convention here (and for Γ^+, θ, etc.) that multiple arguments express a product:

$$\Gamma_{p,q}(z_1, \dots, z_n) = \prod_{1 \leq i \leq n} \Gamma_{p,q}(z_i). \tag{1.16}$$

This satisfies the functional equations

$$\Gamma_{p,q}(qz) = \theta_p(z) \Gamma_{p,q}(z) \tag{1.17}$$

$$\Gamma_{p,q}(pz) = \theta_q(z) \Gamma_{p,q}(z) \tag{1.18}$$

$$\Gamma_{p,q}(pq/z) = \Gamma_{p,q}(z)^{-1}, \tag{1.19}$$

where

$$\theta_p(z) := \prod_{0 \leq i} (1 - p^i z)(1 - p^{i+1}/z) \tag{1.20}$$

is a theta function ($\theta_p(\exp(2\pi i x))$ is doubly quasiperiodic), as well as the "quadratic" functional equations

$$\Gamma_{p,q}(z) = \Gamma_{p,q^2}(z, qz) \tag{1.21}$$

$$\Gamma_{p^2,q^2}(z^2) = \Gamma_{p,q}(z, -z), \tag{1.22}$$

which will be useful below. The special values

$$\Gamma_{p,q^2}(q) = \frac{1}{(q;q^2)} = \frac{(q^2;q^2)}{(q;q)} = (-q;q) \tag{1.23}$$

$$\Gamma_{p,q}(-1) = \frac{(p;p^2)(q;q^2)}{2} \tag{1.24}$$

$$\lim_{x \to 1} (1 - x) \Gamma_{p,q}(x) = \frac{1}{(p;p)(q;q)} \tag{1.25}$$

will arise as well. We will also need a third-order elliptic Gamma function

$$\Gamma^+_{p,q,t}(z) := \prod_{0 \leq i,j,k} (1 - p^{i+1} q^{j+1} t^{k+1}/z)(1 - p^i q^j t^k z), \tag{1.26}$$

with functional equations

$$\Gamma^+_{p,q,t}(tz) = \Gamma_{p,q}(z) \Gamma^+_{p,q,t}(z), \tag{1.27}$$

$$\Gamma^+_{p,q,t}(pqt/z) = \Gamma^+_{p,q,t}(z), \tag{1.28}$$

and so forth. (This will only be used to simplify notation; in all of the cases in which it arises, it will appear only via a ratio that resolves via the first functional equation into a product of usual elliptic Gamma functions.)

The elliptic Selberg integral (introduced as the "elliptic Macdonald-Morris conjecture" in [6], and renamed the "Type II" integral in the follow-up [7]) is the integral with density

$$\begin{aligned}&\Delta^{(n)}(z_1,\dots,z_n;u_0,\dots,u_5;t;p,q)\\&\quad:=\frac{((p;p)(q;q)\Gamma_{p,q}(t))^n}{2^n n!}\prod_{1\le i<j\le n}\frac{\Gamma_{p,q}(tz_i^{\pm 1}z_j^{\pm 1})}{\Gamma_{p,q}(z_i^{\pm 1}z_j^{\pm 1})}\prod_{1\le i\le n}\frac{\prod_{0\le r<6}\Gamma_{p,q}(u_r z_i^{\pm 1})}{\Gamma_{p,q}(z_i^{\pm 2})}\frac{dz_i}{2\pi\sqrt{-1}z_i}\end{aligned} \tag{1.29}$$

with associated evaluation ([15], conjectured in [6])

$$\int_{C^n}\Delta^{(n)}(z_1,\dots,z_n;u_0,\dots,u_5;t;p,q)=\prod_{0\le i<n}\Gamma_{p,q}(t^{i+1})\prod_{0\le r<s<6}\Gamma_{p,q}(t^i u_r u_s), \tag{1.30}$$

where the parameters satisfy the "balancing condition"

$$t^{2n-2}\prod_{0\le r<6}u_r=pq, \tag{1.31}$$

and C is a contour such that $C=C^{-1}$, and C contains the rescaled contour tC together with all points of the form $u_r p^i q^j$. (If one allows suitable disjoint unions of contours, this condition can be satisfied unless $u_r u_s p^i q^j t^k=1$ for some $0\le i,j,k$, $0\le r,s<6$.) By convention, the argument $uz_i^{\pm 1}$ to a function indicates a *pair* of arguments $uz_i, u/z_i$, and similarly for $tz_i^{\pm 1}z_j^{\pm 1}$, etc., so in particular the above integrand is hyperoctahedrally symmetric. This determines a natural normalized linear functional

$$\langle f\rangle^{(n)}_{u_0,\dots,u_5;t;p,q}\propto\int_{C^n}f(z_1,\dots,z_n)\Delta^{(n)}(z_1,\dots,z_n;u_0,\dots,u_5;t;p,q), \tag{1.32}$$

where f is a linear combination of products of hyperoctahedrally symmetric p- and q-elliptic functions such that for some nonnegative integers l_r, m_r, the function

$$f(z_1,\dots,z_n)\prod_{\substack{1\le i\le n\\0\le r\le 5}}\frac{\Gamma_{p,q}(u_r z_i^{\pm 1})}{\Gamma_{p,q}(p^{-l_r}q^{-m_r}u_r z_i^{\pm 1})} \tag{1.33}$$

is holomorphic, and the contour satisifes the conditions appropriate to

$$\Delta^{(n)}(z_1,\dots,z_n;p^{-l_0}q^{-m_0}u_0,\dots,p^{-l_5}q^{-m_5}u_5;t;p,q); \tag{1.34}$$

the integral is normalized so that

$$\langle 1\rangle^{(n)}_{u_0,\dots,u_5;t;p,q}=1. \tag{1.35}$$

Note that if the contour satisfies the conditions for a given choice of l_r, m_r, it satisfies them for all smaller choices, so for a finite linear combination of such functions, one can (generically) choose a contour valid for each term simultaneously, giving linearity. However, the families of functions we consider involve unbounded values of l_0, m_0, and thus one cannot simply fix a single contour for *every* function in the family.

The biorthogonal functions

$$\tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\lambda}}(z_1,\dots,z_n;t_0{:}t_1,t_2,t_3;u_0,u_1;t;p,q) \tag{1.36}$$

of [15, 14] satisfy biorthogonality with respect to this linear functional, i.e.,

$$\langle \tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\lambda}}(z_1,\dots,z_n;t_0{:}t_1,t_2,t_3;u_0,u_1;t;p,q)\,\tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\mu}}(z_1,\dots,z_n;t_0{:}t_1,t_2,t_3;u_1,u_0;t;p,q)\rangle_{t_0,t_1,t_2,t_3,u_0,u_1;t;p,q} \tag{1.37}$$

vanishes unless $\boldsymbol{\lambda}=\boldsymbol{\mu}$. More precisely, they are characterized for generic parameters by this property and the triangularity property that for any partition pairs $\boldsymbol{\kappa}$, $\boldsymbol{\lambda}$, and integers (l,m) with $(l,m)\ge \boldsymbol{\kappa}_1,\boldsymbol{\lambda}_1$ (relative to the product ordering),

$$\lim_{z_i\to(p,q)^{-\boldsymbol{\kappa}_i}t^{i-1}u_0}\prod_{1\le i\le n}\theta(pqz_i^{\pm1}/u_0;p,q)_{l,m}\,\tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\lambda}}(z_1,\dots,z_n;t_0{:}t_1,t_2,t_3;u_0,u_1;t;p,q)=0 \tag{1.38}$$

unless $\boldsymbol{\kappa}\subset\boldsymbol{\lambda}$; here and below, $(p,q)^{(l,m)}:=p^lq^m$. The biorthogonal functions are normalized by taking

$$\tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\lambda}}(\dots,t^{n-i}t_0,\dots;t_0{:}t_1,t_2,t_3;u_0,u_1;t;p,q)=1; \tag{1.39}$$

though this breaks the symmetry between the four t_r parameters (only mildly: the required changes in normalization have explicit product formulas), it makes the biorthogonal function with index $(0,\mu)$ p-elliptic in every parameter.

We will also need higher order versions of the elliptic Selberg integral; we define

$$I\!I^{(m)}_n(u_0,\dots,u_{2m+5};t;p,q):=\int_{C^n}\Delta^{(n)}(z_1,\dots,z_n;u_0,\dots,u_{2m+5};t;p,q), \tag{1.40}$$

subject to the balancing condition

$$t^{2n-2}\prod_{0\le r<2m+6}u_r=(pq)^{m+1}, \tag{1.41}$$

in which the density is obtained from the original density ($m=0$) by replacing

$$\prod_{0\le r<6}\Gamma_{p,q}(u_rz_i^{\pm1})\mapsto\prod_{0\le r<2m+6}\Gamma_{p,q}(u_rz_i^{\pm1}), \tag{1.42}$$

and the contour condition is extended in the obvious way. In particular, if $u_{2m+4}u_{2m+5}=pq$, then the reflection equation for $\Gamma_{p,q}$ causes the two corresponding factors to cancel, reducing m by 1. When $n=1$, the higher-order elliptic Selberg integral is essentially independent of t, apart from the factor $\Gamma_{p,q}(t)$; we thus define the higher-order elliptic beta integral [21] by

$$I^{(m)}(u_0,\dots,u_{2m+5};p,q):=\Gamma_{p,q}(t)^{-1}I\!I^{(m)}_1(u_0,\dots,u_{2m+5};t;p,q); \tag{1.43}$$

note that the constraint that the contour C contains tC is irrelevant in this case.

When $m=1$, the elliptic Selberg integral satisfies an important transformation (a special case of [15, Thm. 9.7]), namely that

$$I\!I_n(u_0,\dots,u_7;t;p,q)=I\!I_n(u_0/v,u_1/v,u_2/v,u_3/v,u_4v,u_5v,u_6v,u_7v)\prod_{\substack{0\le i<n\\0\le r<s<4}}\Gamma_{p,q}(t^iu_ru_s,t^iu_{r+4}u_{s+4}), \tag{1.44}$$

where $v^2=\frac{pqt^{1-n}}{u_4u_5u_6u_7}=\frac{u_0u_1u_2u_3}{pqt^{1-n}}=\sqrt{\frac{u_0u_1u_2u_3}{u_4u_5u_6u_7}}$. Together with permutations of the parameters, this generates the Weyl group of type E_7. We note the following special case, which will arise repeatedly in Section 5 below.

Proposition 1.1. *Define a function*

$$F_n(t_0,t_1,t_2,t_3;a,b;t;p,q) = I\!I_n(a^{\pm 1/2}t_0, a^{\pm 1/2}t_1, a^{\pm 1/2}t_2, (b^2a)^{\pm 1/2}t_3;t;p,q) \prod_{\substack{0\le i<n\\ 0\le r<s<3}} \frac{\Gamma_{p,q}(t^ibt_rt_s, t^iabt_rt_s)}{\Gamma_{p,q}(t^it_rt_s, t^it_rt_s/a)}, \tag{1.45}$$

subject to the balancing condition $t^{n-1}t_0t_1t_2t_3 = pq$. Then F_n is invariant under permuting t_0,t_1,t_2,t_3 and under swapping a and b.

Remark. This function satisfies additional transformations

$$F_n(t_0,t_1,t_2,t_3;a,b;t;p,q) = F_n(t_0,t_1,t_2,t_3;ab,1/b;t;p,q) \prod_{0\le i<n, 0\le r<s<4} \Gamma_{p,q}(t^it_rt_sb) \tag{1.46}$$

and

$$F_n(t_0,t_1,t_2,t_3;a,b;t;p,q) = F_n(\gamma/t_0,\gamma/t_1,\gamma/t_2,\gamma/t_3;a,b;t;p,q) \prod_{\substack{0\le i<n\\ 0\le r<4}} \Gamma_{p,q}(t^it_r^2) \tag{1.47}$$

where $\gamma = (t^{1-n}pq)^{1/2}$. These transformations generate a Weyl group $B_3 \times G_2$, and in much the same way as the general order 1 elliptic Selberg integral satisfies a formal E_8 symmetry (see discussion in [15, §9]), this group formally extends to an action of $F_4 \times G_2$.

The factors

$$\Delta^0_{\boldsymbol\lambda}(a|b_0,\dots,b_{n-1};t;p,q) \quad \text{and} \quad \Delta_{\boldsymbol\lambda}(a|b_0,\dots,b_{n-1};t;p,q) \tag{1.48}$$

that appear below are certain multivariate q-symbols (see the introduction of [15]). The first is defined by

$$\Delta^0_{\boldsymbol\lambda}(a|b_0,\dots,b_{n-1};t;p,q) = \prod_{0\le r<n} \frac{\mathcal{C}^0_{\boldsymbol\lambda}(b_r;t;p,q)}{\mathcal{C}^0_{\boldsymbol\lambda}(pqa/b_r;t;p,q)}, \tag{1.49}$$

where

$$\mathcal{C}^0_{\boldsymbol\lambda}(x;t;p,q) := \prod_{1\le i} \theta(t^{1-i}x;p,q)_{\boldsymbol\lambda_i}, \tag{1.50}$$

and

$$\theta(x;p,q)_{l,m} := \prod_{0\le j<l} \theta_q(p^jx) \prod_{0\le j<m} \theta_p(q^jx). \tag{1.51}$$

Note that

$$\Delta^0_{\lambda,\mu}(a|b_0,\dots,b_{n-1};t;p,q) = \Delta^0_{\lambda,0}(a|b_0,\dots,b_{n-1};t;p,q)\Delta^0_{0,\mu}(a|b_0,\dots,b_{n-1};t;p,q), \tag{1.52}$$

and if $n=2m$, $\prod_{0\le r<2m} b_r = (pqa)^m$, then both factors are elliptic subject to this constraint; i.e.,

$$\Delta^0_{0,\mu}(a|b_0,\dots,b_{2m-1};t;p,q) \tag{1.53}$$

is invariant under shifting the parameters by integer powers of p such that the balancing condition remains satisfied.

The other Δ-symbol is more complicated:

$$\Delta_{\boldsymbol\lambda}(a|b_0,\dots,b_{n-1};t;p,q) := \Delta^0_{\boldsymbol\lambda}(a|b_0,\dots,b_{n-1};t;p,q) \frac{\mathcal{C}^0_{2\boldsymbol\lambda^2}(pqa;t;p,q)}{\mathcal{C}^-_{\boldsymbol\lambda}(pq,t;t;p,q)\,\mathcal{C}^+_{\boldsymbol\lambda}(a,pqa/t;t;p,q)} \tag{1.54}$$

where

$$\mathcal{C}^-_{\boldsymbol\lambda}(x;t;p,q) := \prod_{1\le i\le j} \frac{\theta(t^{j-i}x;p,q)_{\boldsymbol\lambda_i-\boldsymbol\lambda_{j+1}}}{\theta(t^{j-i}x;p,q)_{\boldsymbol\lambda_i-\boldsymbol\lambda_j}} \tag{1.55}$$

$$\mathcal{C}^+_{\boldsymbol\lambda}(x;t;p,q) := \prod_{1\le i\le j} \frac{\theta(t^{2-i-j}x;p,q)_{\boldsymbol\lambda_i+\boldsymbol\lambda_j}}{\theta(t^{2-i-j}x;p,q)_{\boldsymbol\lambda_i+\boldsymbol\lambda_{j+1}}}. \tag{1.56}$$

The key property of $\Delta_{\boldsymbol\lambda}$ is that the $\boldsymbol\lambda$-dependent factor of the residue of the elliptic Selberg integrand $\Delta^{(n)}$ at the point $(\dots,(p,q)^{\boldsymbol\lambda}t^{n-i}u_0,\dots)$ is

$$\Delta_{\boldsymbol\lambda}(t^{2n-2}u_0^2|t^n,t^{n-1}u_0u_1,\dots,t^{n-1}u_0u_{2m+5};t;p,q). \tag{1.57}$$

The corresponding balancing condition to ensure ellipticity is, for $n=2m$, that $\prod_{0\le r<2m} b_r = (t/pq)(pqa)^{m-1}$.

In many respects, the most natural elliptic analogue of the Macdonald polynomials is the interpolation functions, a special case of the biorthogonal functions given by

$$\mathcal{R}^{*(n)}_{\boldsymbol\lambda}(;t_0,u_0;t;p,q) = \Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0/u_0|t^{n-1}t_0t_1,t_0/t_1;t;p,q)\,\tilde{\mathcal{R}}^{(n)}_{\boldsymbol\lambda}(;t_1{:}t_0,t_2,t_3;u_0,t^{1-n}/t_0;t;p,q) \tag{1.58}$$

with $t^{n-1}t_1t_2t_3u_0=pq$. Note that the left-hand side is independent of the remaining degrees of freedom. The key property of the interpolation functions is that

$$\mathcal{R}^{*(n)}_{\boldsymbol\lambda}(\dots,(p,q)^{\boldsymbol\mu_i}t^{n-i}a,\dots;a,b;t;p,q)=0 \tag{1.59}$$

unless $\boldsymbol\lambda\subset\boldsymbol\mu$ ([15, Cor. 8.12]); this property and the triangularity property are related by a complementation symmetry, and together determine the interpolation function up to normalization, which is determined by

$$\mathcal{R}^{*(n)}_{\boldsymbol\lambda}(\dots,t^{n-i}v,\dots;a,b;t;p,q) = \Delta^0_{\boldsymbol\lambda}(t^{n-1}a/b|t^{n-1}av,a/v;t;p,q). \tag{1.60}$$

The interpolation functions play a special role in the theory of the elliptic biorthogonal functions, as certain connection coefficients between biorthogonal functions with slightly different parameters can be expressed via values of interpolation functions at partitions [14, Cor. 5.7]. As a special case, any biorthogonal function can be expanded as a linear combination of interpolation functions in which the coefficients are themselves values of interpolation functions [14, Defn. 12 and Thm. 5.3].

These values of interpolation functions appear frequently enough to merit their own notation: we define

$$\binom{\boldsymbol\lambda}{\boldsymbol\mu}_{[a,b];t;p,q} := \Delta_{\boldsymbol\mu}(a/b|t^n,1/b;t;p,q)\,\mathcal{R}^{*(n)}_{\boldsymbol\mu}(\dots,\sqrt{a}(p,q)^{\boldsymbol\lambda_i}t^{1-i},\dots;t^{1-n}\sqrt{a},b/\sqrt{a};t;p,q); \tag{1.61}$$

this is independent of the choice of square root, and factors as

$$\binom{\lambda,\kappa}{\mu,\nu}_{[a,b];t;p,q} = \binom{\lambda}{\mu}_{[a,b];p,t;q}\binom{\kappa}{\nu}_{[a,b];q,t;p} \tag{1.62}$$

where the first factor is q-elliptic in a, b, p, and t, and imilarly for the second factor. We also use the alternate normalization of [14], which in the p,q-symmetric version reads

$$\left\langle{\boldsymbol\lambda \atop \boldsymbol\mu}\right\rangle_{[a,b](v_1,\dots,v_k);t;p,q} := \frac{\Delta^0_{\boldsymbol\lambda}(a|b,v_1,\dots,v_k;t;p,q)}{\Delta^0_{\boldsymbol\mu}(a/b|1/b,v_1,\dots,v_k;t;p,q)}\binom{\boldsymbol\lambda}{\boldsymbol\mu}_{[a,b];t;p,q}. \tag{1.63}$$

The binomial coefficients so normalized are products of elliptic functions if $k=3$, $bv_1v_2v_3=(pqa)^2$.

2 Skew interpolation functions

Consider the following generalized elliptic hypergeometric sum:

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0,\dots,v_{2n-1}];a,b;t;p,q) := \sum_{\boldsymbol{\kappa}\subset\boldsymbol{\mu}\subset\boldsymbol{\lambda}} \left\langle \begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[a/b,ab/pq];t;p,q} \left\langle \begin{matrix}\boldsymbol{\mu}\\ \boldsymbol{\kappa}\end{matrix}\right\rangle_{[pq/b^2,pq\prod_{0\le r<2n} v_r/ab];t;p,q} \times \Delta^0_{\boldsymbol{\mu}}(pq/b^2|pq/bv_0,pq/bv_1,\dots,pq/bv_{2n-1};t;p,q); \tag{2.1}$$

as the notation suggests, this will turn out to be our desired skew version of the interpolation functions. Note that each term in the rescaled sum

$$\hat{\mathcal{R}}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0,\dots,v_{2n-1}];a,b;t;p,q) := \frac{\Delta^0_{\boldsymbol{\kappa}}(a/b\prod_{0\le r<2n} v_r|ab/pq\prod_{0\le r<2n} v_r;t;p,q)}{\Delta^0_{\boldsymbol{\lambda}}(a/b|ab/pq;t;p,q)} \mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0,\dots,v_{2n-1}];a,b;t;p,q) \tag{2.2}$$

is the product of p-abelian and q-abelian factors, so the same applies to this rescaled sum; however, the rescaling introduces unfortunate poles, so we will prefer to use the not-quite-elliptic form unless that would introduce complicated factors from quasiperiodicity. This is a generalized elliptic hypergeometric sum in the same sense as the identities of [14]; in particular, it includes the following very-well-poised, balanced, and terminating multivariate elliptic hypergeometric series as a special case:

$$\hat{\mathcal{R}}^*_{(l,m)^n/0}([v_0,\dots,v_{2k-1}];a,b;t;p,q) = \sum_{\boldsymbol{\mu}\subset(l,m)^n} \Delta_{\boldsymbol{\mu}}(pq/b^2|t^n,p^{-l}q^{-m},p^lq^ma/t^{n-1}b,pq/bv_0,pq/bv_1,\dots,pq/bv_{2k-1},pq\prod_{0\le r<2k} v_r/ab;t;p,q). \tag{2.3}$$

(Such sums arise as limiting cases of order $k-1$ elliptic Selberg integrals via residue calculus.) We note that the skew interpolation function is invariant under permutations of its arguments, as well as under insertion or deletion of pairs $x, 1/x$. (The last statement follows from the fact that

$$\Delta^0_{\boldsymbol{\mu}}(pq/b^2|pq/bx,pqx/b;t;p,q) = 1, \tag{2.4}$$

which in turn is immediate from the definition.) In particular, the arguments are not directly arguments of interpolation functions, but play a more plethystic role. Roughly speaking, this corresponds to the plethystic substitution

$$p_k \mapsto \sum_{0\le r<2n} \frac{v_r^k - v_r^{-k}}{t^{k/2}-t^{-k/2}} \tag{2.5}$$

at the trigonometric level, so that an ordinary argument corresponds to a pair $t^{1/2}x, t^{1/2}/x$ of plethystic arguments. (Compare Theorem 2.5 below.)

The two main identities of [14] both involved sums of this form, and thus one has the following.

Proposition 2.1. *[14, Cor. 4.3] With no arguments, the skew interpolation function is a delta function:*

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([];a,b;t;p,q) = \delta_{\boldsymbol{\lambda}\boldsymbol{\kappa}}. \tag{2.6}$$

Proposition 2.2. *[14, Thm. 4.1] With two arguments, the skew interpolation function is an elliptic binomial coefficient:*

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0,v_1];a,b;t;p,q) = \left\langle \begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\kappa}\end{matrix} \right\rangle_{[a/b,v_0v_1](a/v_0,a/v_1);t;p,q}. \tag{2.7}$$

Remark. When $v_0v_1 = 1$, so we can eliminate the two arguments to the skew interpolation function, the right-hand side specializes to a delta function as required.

Proposition 2.3. *[14, Thm. 4.9, Cor. 4.11] With four arguments, the skew interpolation function has the alternate expressions*

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0,v_1,v_2,v_3];a,b;t;p,q) = \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix} \right\rangle_{[a/b,v_0v_1](a/v_0,a/v_1);t;p,q} \left\langle \begin{matrix}\boldsymbol{\mu}\\ \boldsymbol{\kappa}\end{matrix} \right\rangle_{[a/v_0v_1b,v_2v_3](a/v_0v_1v_2,a/v_0v_1v_3);t;p,q} \tag{2.8}$$

and

$$\begin{aligned}\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0,v_1,v_2,v_3];a,b;t;p,q) = {}& \frac{\Delta^0_{\boldsymbol{\lambda}}(a/b|a/v_0,a/v_1,a/v_2,a/v_3;t;p,q)}{\Delta^0_{\boldsymbol{\kappa}}(a/bV|av_0/V,av_1/V,av_2/V,av_3/V;t;p,q)} \\ & \times \sum_{\boldsymbol{\mu}} \frac{\left\langle \begin{smallmatrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{smallmatrix} \right\rangle_{[a/b,pqV/ab];t;p,q} \left\langle \begin{smallmatrix}\boldsymbol{\mu}\\ \boldsymbol{\kappa}\end{smallmatrix} \right\rangle_{[a^2/pqV,ab/pq];t;p,q}}{\Delta^0_{\boldsymbol{\mu}}(a^2/pqV|a/v_0,a/v_1,a/v_2,a/v_3;t;p,q)},\end{aligned} \tag{2.9}$$

where $V = v_0v_1v_2v_3$.

In equation (2.8), the binomial coefficients can be expressed in skew interpolation functions, giving

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0,v_1,v_2,v_3];a,b;t;p,q) = \sum_{\boldsymbol{\mu}} \mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([v_0,v_1];a,b;t;p,q)\,\mathcal{R}^*_{\boldsymbol{\mu}/\boldsymbol{\kappa}}([v_2,v_3];a/v_0v_1,b;t;p,q). \tag{2.10}$$

This generalizes considerably.

Proposition 2.4. *The skew interpolation functions satisfy the identity*

$$\begin{aligned}\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0,\dots,v_{2k-1},w_0,\dots,w_{2l-1}];a,b;t;p,q) = {}& \sum_{\boldsymbol{\mu}} \mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([v_0,\dots,v_{2k-1}];a,b;t;p,q) \\ & \times \mathcal{R}^*_{\boldsymbol{\mu}/\boldsymbol{\kappa}}([w_0,\dots,w_{2l-1}];a/v_0\cdots v_{2k-1},b;t;p,q).\end{aligned} \tag{2.11}$$

Proof. If we expand the skew interpolation functions on the right via the definition, the inner sum over $\boldsymbol{\mu}$ is itself a skew interpolation function with no arguments, and thus the inner sum collapses as required. ☐

Thus to justify the name "skew interpolation function", it remains only to show that when $\boldsymbol{\kappa} = 0$, we obtain (a generalization of) the usual interpolation function.

Theorem 2.5. *The interpolation functions have the expression*

$$\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(z_1,\dots,z_n;a,b;t;p,q) = \Delta^0_{\boldsymbol{\lambda}}(t^{n-1}a/b|pqa/tb;t;p,q)\,\mathcal{R}^*_{\boldsymbol{\lambda}/0}([t^{1/2}z_1^{\pm 1},\dots,t^{1/2}z_n^{\pm 1}];t^{n-1/2}a,t^{1/2}b;t;p,q). \tag{2.12}$$

Proof. By the connection coefficient identity [14, Cor. 4.14], we can write

$$\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(z_1,\dots,z_n;a,b;t;p,q) = \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[t^{n-1}a/b,t^n ab/pq](pqa/tb);t;p,q} \mathcal{R}^{*(n)}_{\boldsymbol{\mu}}(z_1,\dots,z_n;pq/t^n b,b;t;p,q). \tag{2.13}$$

But the new interpolation functions are of “Cauchy” type, so by [14, Prop. 3.9],

$$\mathcal{R}^{*(n)}_{\boldsymbol{\mu}}(z_1,\dots,z_n;pq/t^n b,b;t;p,q) = \Delta^0_{\boldsymbol{\mu}}(pq/tb^2|pqz_1^{\pm 1}/tb,\dots,pqz_n^{\pm 1}/tb;t;p,q) \tag{2.14}$$

as required. ☐

Remark 1. This can also be proved by induction via the branching rule [14, Thm. 4.16]. Similarly, we find that the coefficients of (1.7) are given by

$$\begin{aligned}&\mathcal{R}^{*(m,n)}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(z_1,\dots,z_m;a,b;t;p,q)\\ &\quad= \frac{\Delta^0_{\boldsymbol{\lambda}}(t^{n+m-1}a/b|pqa/tb;t;p,q)}{\Delta^0_{\boldsymbol{\mu}}(t^{n-1}a/b|pqa/tb;t;p,q)}\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([t^{1/2}z_1^{\pm 1},\dots,t^{1/2}z_m^{\pm 1}];t^{n+m-1/2}a,t^{1/2}b;t;p,q)\end{aligned} \tag{2.15}$$

Remark 2. Thus ordinary interpolation functions correspond to the case that the arguments multiply pairwise to t; similarly, the skew interpolation functions of [5] correspond to the special case in which the arguments multiply pairwise to some general, but fixed, r.

Remark 3. The inverse expansion:

$$\Delta^0_{\boldsymbol{\lambda}}(pq/tb^2|pqz_1^{\pm 1}/tb,\dots,pqz_n^{\pm 1}/tb;t;p,q) = \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[pq/tb^2,pq/t^n ab](pqa/tb);t;p,q} \mathcal{R}^{*(n)}_{\boldsymbol{\mu}}(z_1,\dots,z_n;a,b;t;p,q) \tag{2.16}$$

holds even if $\ell(\boldsymbol{\lambda}) > n$ (assuming generic parameters). Indeed, if k is sufficiently large, so that $n+k \ge \ell(\boldsymbol{\lambda})$, then one may set $z_{n+i} = t^{-i}a$ in

$$\begin{aligned}&\Delta^0_{\boldsymbol{\lambda}}(pq/tb^2|pqz_1^{\pm 1}/tb,\dots,pqz_{n+k}^{\pm 1}/tb;t;p,q)\\ &\qquad= \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[pq/tb^2,pq/t^n ab](pqt^{-k}a/tb);t;p,q} \mathcal{R}^{*(n+k)}_{\boldsymbol{\mu}}(z_1,\dots,z_{n+k};t^{-k}a,b;t;p,q)\end{aligned} \tag{2.17}$$

to obtain the desired result. This will be useful in the sequel, as products of this form satisfy a number of useful identities. For convenience in notation, we will use the product expression (2.14) to extend the Cauchy-type interpolation functions to the case that the indexing partition has more than n parts, as the above considerations eliminate most of the dangers in such an extension.

With this in mind, we refer to the functions $\mathcal{R}^*_{\boldsymbol{\lambda}/0}$ as *lifted interpolation functions*; these seem to be about as close as one can hope to get to an elliptic analogue of the lifted interpolation polynomials of [13, §6]. These functions have a somewhat surprising additional symmetry.

Proposition 2.6. *The lifted interpolation function* $\mathcal{R}^*_{\boldsymbol{\lambda}/0}([v_0,\dots,v_{2n-1}];a,b;t;p,q)$ *is invariant under permutations of the* $2n+1$ *values*

$$v_0,\dots,v_{2n-1},a/\prod_{0\le r<2n} v_r. \tag{2.18}$$

Proof. Since

$$\left\langle \begin{matrix} \boldsymbol{\mu} \\ 0 \end{matrix} \right\rangle_{[a,b];t;p,q} = \Delta^0_{\boldsymbol{\mu}}(a|b;t;p,q), \tag{2.19}$$

we have

$$\begin{aligned}\mathcal{R}^*_{\boldsymbol{\lambda}/0}([v_0,\dots,v_{2n-1}];a,b;t;p,q) = \sum_{\boldsymbol{\mu}} &\left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[a/b,ab/pq];t;p,q} \\ &\times \Delta^0_{\boldsymbol{\mu}}(pq/b^2|pq/bv_0,pq/bv_1,\dots,pq/bv_{2n-1},pq\prod_{0\le r<2n} v_r/ab;t;p,q),\end{aligned} \tag{2.20}$$

which manifestly has the stated symmetry. ∎

It follows that the connection coefficient formula of [14] extends, and in a particularly nice form.

Corollary 2.7. *One has the identity*

$$\mathcal{R}^*_{\boldsymbol{\lambda}/0}([v_0,\dots,v_{2n-1}];a,b;t;p,q) = \sum_{\boldsymbol{\kappa}} \mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([a/V,V/a'];a,b;t;p,q)\,\mathcal{R}^*_{\boldsymbol{\kappa}/0}([v_0,\dots,v_{2n-1}];a',b;t;p,q), \tag{2.21}$$

where $V = \prod_{0\le r<2n} v_r$.

Proof. Indeed, this reduces to showing

$$\mathcal{R}^*_{\boldsymbol{\lambda}/0}([a/V,V/a',v_0,\dots,v_{2n-1}];a,b;t;p,q) = \mathcal{R}^*_{\boldsymbol{\lambda}/0}([v_0,\dots,v_{2n-1}];a,b;t;p,q), \tag{2.22}$$

and this is simply deletion of the pair $V/a', a'/V$ from the left-hand side, after applying Proposition 2.6. ∎

The relation of skew interpolation functions to the binomial coefficients means that we can expect most symmetries of the latter to extend. We begin with duality, which breaks the symmetry between p and q, but will be useful in the sequel. Here we can simply apply the symmetry term-by-term in the definition.

Proposition 2.8. *[14, Cor. 4.4]*

$$\mathcal{R}^*_{(0,\lambda)/(0,\kappa)}([v_0,\dots,v_{2k-1}];a,b;t;p,q) = \mathcal{R}^*_{(0,\lambda')/(0,\kappa')}([v_0,\dots,v_{2k-1}];a,b/qt;1/q;p,1/t) \tag{2.23}$$

The other symmetries do respect the p, q symmetry, but lead to unpleasant scale factors since the skew interpolation functions are not quite elliptic, so we use the $\hat{\mathcal{R}}^*$ variant. In particular, this allows one to prove identities by factoring into p-elliptic and q-elliptic factors, then using ellipticity to restore symmetry before multiplying the identities back together.

Proposition 2.9. *[14, (4.10)] If* $\ell(\boldsymbol{\lambda}), \ell(\boldsymbol{\kappa}) \le n$, *then*

$$\begin{aligned}&\hat{\mathcal{R}}^*_{(l,m)^n+\boldsymbol{\lambda}/(l,m)^n+\boldsymbol{\kappa}}([\dots,v_r,\dots];a,b;t;p,q) \\ &\qquad= \Delta^0_{(l,m)^n}(pq/b^2|\dots,pq/bv_r,\dots,Qa/t^{n-1}b,p^2q^2t^{n-1}V/Qab;t;p,q) \\ &\qquad\quad\times\left(\frac{\Delta^0_{\boldsymbol{\lambda}}(Q^2a/b|pqt^{n-1}Q,Q^2a/t^{n-1}b,ab/pq,p^2q^2Q/b^2;t;p,q)}{\Delta^0_{\boldsymbol{\kappa}}(Q^2a/Vb|pqt^{n-1}Q,Q^2a/t^{n-1}bV,ab/pqV,p^2q^2Q/b^2;t;p,q)}\right) \\ &\qquad\quad\times\hat{\mathcal{R}}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([\dots,v_r,\dots];Qa,b/Q;t;p,q),\end{aligned} \tag{2.24}$$

where $V = \prod_r v_r$ *and* $Q = p^l q^m$.

Proposition 2.10. *[14, Cor. 4.6] If $\boldsymbol{\lambda}_1, \boldsymbol{\kappa}_1 \le (l,m)$, then*

$$\begin{aligned}
&\hat{\mathcal{R}}^*_{(l,m)^n\cdot\boldsymbol{\lambda}/(l,m)^n\cdot\boldsymbol{\kappa}}([\dots,v_r,\dots];a,b;t;p,q)\\
&\qquad = \Delta^0_{(l,m)^n}(pq/b^2|\dots,pq/bv_r,\dots,Qa/t^{n-1}b,p^2q^2t^{n-1}V/Qab;t;p,q)\\
&\qquad\quad \times \left(\frac{\Delta^0_{\boldsymbol{\lambda}}(a/t^{2n}b|pq/Qt^{n+1},Qa/t^{2n-1}b,ab/pq,p^2q^2/t^nb^2;t;p,q)}{\Delta^0_{\boldsymbol{\kappa}}(a/t^{2n}Vb|pq/Qt^{n+1},Qa/t^{2n-1}bV,ab/pqV,p^2q^2/t^nb^2;t;p,q)}\right)\\
&\qquad\quad \times \hat{\mathcal{R}}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([\dots,v_r,\dots];t^{-n}a,t^nb;t;p,q). \qquad (2.25)
\end{aligned}$$

Proposition 2.11. *[14, Cor. 4.7] If $\boldsymbol{\lambda},\boldsymbol{\kappa} \subset (l,m)^n$, then*

$$\begin{aligned}
&\hat{\mathcal{R}}^*_{(l,m)^n-\boldsymbol{\kappa}/(l,m)^n-\boldsymbol{\lambda}}([\dots,v_r,\dots];a,b;t;p,q)\\
&\qquad = \Delta^0_{(l,m)^n}(pq/b^2|\dots,pq/bv_r,\dots,Qa/t^{n-1}b,p^2q^2t^{n-1}V/Qab;t;p,q)\\
&\qquad\quad \times \left(\frac{\Delta_{\boldsymbol{\lambda}}(t^{2n-2}bV/Q^2a|t^n,1/Q,t^{n-1}b^2/pqQ,pqV/ab;t;p,q)}{\Delta_{\boldsymbol{\kappa}}(t^{2n-2}b/Q^2a|t^n,1/Q,t^{n-1}b^2/pqQ,pq/ab;t;p,q)}\right)\\
&\qquad\quad \times \hat{\mathcal{R}}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([\dots,v_r,\dots];pqVt^{n-1}/Qa,pqQ/t^{n-1}b;t;p,q). \qquad (2.26)
\end{aligned}$$

The above symmetries each follow by applying the corresponding symmetries of elliptic binomial coefficients and Δ symbols to the definition of the skew interpolation functions. There is also an analogue of [14, Cor. 4.8], but this is more subtle. We give this in a fairly general form, for ease of induction and later application.

Theorem 2.12. *Suppose the parameters $v_0,\dots,v_{2k-1}$ can be ordered in such a way that $v_{2r}v_{2r+1} = t^{n_r}$ with $n_r \in \mathbb{Z}_{\ge 0}$, $0 \le r < k$, and let l, m, n, n' be nonnegative integers with $n' = n + \sum_r n_r$ and $\boldsymbol{\lambda}_1, \boldsymbol{\kappa}_1 \le (l,m)$. Then*

$$\begin{aligned}
&\hat{\mathcal{R}}^*_{(l,m)^{n'}\cdot\boldsymbol{\lambda}/(l,m)^n\cdot\boldsymbol{\kappa}}([v_0,\dots,v_{2k-1}];a,b;t;p,q)\\
&\qquad = \frac{\prod_{0\le r<k}\Delta^0_{(l,m)^{n_r}}(a/bt^{n'-n_r}|av_{2r}/t^{n'},av_{2r+1}/t^{n'};t;p,q)}{\Delta^0_{(l,m)^{n'-n}}(a/bt^n|ab/pq,pqa/t^{n'+n}b;t;p,q)}\\
&\qquad\quad \times \left(\frac{\Delta_{\boldsymbol{\kappa}}(a/t^{n'+n}b|p^{-l}q^{-m},p^lq^ma/t^{n'-1}b,ab/pqt^{n'},pqt^{n'-n}/ab;t;p,q)}{\Delta_{\boldsymbol{\lambda}}(a/t^{2n'}b|p^{-l}q^{-m},p^lq^ma/t^{n'-1}b,ab/pqt^{n'},pq/ab;t;p,q)}\right)\\
&\qquad\quad \times \hat{\mathcal{R}}^*_{\boldsymbol{\kappa}/\boldsymbol{\lambda}}([v_0,\dots,v_{2k-1}];pq/t^nb,pqt^{n'}/a;t;p,q) \qquad (2.27)
\end{aligned}$$

Proof. When $k=1$, this follows immediately from Corollaries 4.6 and 4.8 of [14]. (Corollary 4.8 corresponds to the case $n=0$, $k=1$, and Corollary 4.6 allows one to extend this to $n>0$.) We then proceed by induction on k. One first notes that

$$\begin{aligned}
&\hat{\mathcal{R}}^*_{(l,m)^{n'}\cdot\boldsymbol{\lambda}/(l,m)^n\cdot\boldsymbol{\kappa}}([v_0,\dots,v_{2k-1}];a,b;t;p,q)\\
&\qquad = \sum_{\boldsymbol{\mu}} \hat{\mathcal{R}}^*_{(l,m)^{n'}\cdot\boldsymbol{\lambda}/\boldsymbol{\mu}}([v_0,v_1];a,b;t;p,q)\hat{\mathcal{R}}^*_{\boldsymbol{\mu}/(l,m)^n\cdot\boldsymbol{\kappa}}([v_2,\dots,v_{2k-1}];a/t^{n_0},b;t;p,q). \qquad (2.28)
\end{aligned}$$

The key observation is that the first factor is

$$\left\langle \begin{matrix}(l,m)^{n'}\cdot\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[a/b,t^{n_0}](a/v_0,a/v_1,p^2q^2/b^2);t;p,q}, \qquad (2.29)$$

which vanishes unless

$$\boldsymbol{\mu}_i \le ((l,m)^{n'} \cdot \boldsymbol{\lambda})_i \le \boldsymbol{\mu}_{i-n_0}. \tag{2.30}$$

In particular, for $1 \le i \le n'$, $\boldsymbol{\mu}_i \le (l,m)$, while for $1 \le i \le n'-n_0$, $\boldsymbol{\mu}_i \ge (l,m)$. We can thus rewrite the sum as

$$\sum_{\boldsymbol{\nu}} \hat{\mathcal{R}}^*_{(l,m)^{n'}\cdot\boldsymbol{\lambda}/(l,m)^{n'-n_0}\cdot\boldsymbol{\nu}}([v_0,v_1];a,b;t;p,q)\hat{\mathcal{R}}^*_{(l,m)^{n'-n_0}\cdot\boldsymbol{\nu}/(l,m)^n\cdot\boldsymbol{\kappa}}([v_2,\dots,v_{2k-1}];a/t^{n_0},b;t;p,q). \tag{2.31}$$

The result follows by applying the symmetry to each factor and simplifying. ☐

Dually, one has the following identity.

Corollary 2.13. *Suppose the parameters $v_0,\dots,v_{2k-1}$ can be ordered in such a way that $v_{2r}v_{2r+1} = Q_r^{-1}$, $Q_r := p^{l_r}q^{m_r}$, with $l_r, m_r \in \mathbb{Z}_{\ge 0}$ for $0 \le r < k$, and let l, l', m, m', n, be nonnegative integers with $l' = l+\sum_r l_r$, $m' = m + \sum_r m_r$, $\ell(\boldsymbol{\lambda}), \ell(\boldsymbol{\kappa}) \le n$. Then*

$$\begin{aligned}
&\hat{\mathcal{R}}^*_{(l',m')^n+\boldsymbol{\lambda}/(l,m)^n+\boldsymbol{\kappa}}([v_0,\dots,v_{2k-1}];a,b;t;p,q)\\
&\qquad = \frac{\prod_{0\le r<k}\Delta^0_{(l_r,m_r)^n}(Q'a/Q_rb|Q'a/Q_rv_{2r},Q'a/Q_rv_{2r+1};t;p,q)}{\Delta^0_{(l'-l,m'-m)^n}(Qa/b|ab/pq,pqQQ'a/b;t;p,q)}\\
&\qquad\quad \times \left(\frac{\Delta_{\boldsymbol{\kappa}}(aQQ'/b|t^n,atQ'/t^nb,abQ'/pq,pqQ/Q'ab;t;p,q)}{\Delta_{\boldsymbol{\lambda}}(aQ'^2/b|t^n,atQ'/t^nb,abQ'/pq,pq/ab;t;p,q)}\right)\\
&\qquad\quad \times \hat{\mathcal{R}}^*_{\boldsymbol{\kappa}/\boldsymbol{\lambda}}([v_0,\dots,v_{2k-1}];pqQ/b,pq/Q'a;t;p,q),
\end{aligned} \tag{2.32}$$

where $Q = p^lq^m$, $Q' = p^{l'}q^{m'}$.

If $\boldsymbol{\lambda} = 0$ in the first identity, one can apply complementation to obtain a relation between $\hat{\mathcal{R}}^*_{\boldsymbol{\kappa}/0}$ and $\hat{\mathcal{R}}^*_{(l,m)^n-\boldsymbol{\kappa}/0}$; the constraint on the arguments causes both of these to be ordinary interpolation functions in n variables, and this is just the usual complementation symmetry of such functions. (In contrast, in the corresponding special case of the corollary, the lifted interpolation functions are not simply ordinary interpolation functions.) Particularly interesting is the case that both $\boldsymbol{\kappa}$ and its complement are rectangles, since then the identity is a transformation of more classically hypergeometric sums (under the hypotheses of Theorem 2.12):

$$\begin{aligned}
&\hat{\mathcal{R}}^*_{(l,m)^n/0}([v_0,\dots,v_{2k-1}];a,b;t;p,q)\\
&\qquad \propto \prod_{0\le i<2k}\frac{\Gamma^+_{p,q,t}(b/v_i,(Q(pqt/b))/v_i,(t^{-n-n'}at)/v_i,(pq/Qt^{-n-n'}a)/v_i)}{\Gamma^+_{p,q,t}(bv_i,Q(pqt/b)v_i,t^{-n-n'}atv_i,(pqt/Qt^{-n-n'}at)v_i)}\\
&\qquad\quad \times \hat{\mathcal{R}}^*_{(l,m)^{n'}/0}([v_0,\dots,v_{2k-1}];t^{n+n'-1}b/Q,Qa/t^{n+n'-1};t;p,q),
\end{aligned} \tag{2.33}$$

where the constant of proportionality can be determined from the case $k=1$, when both lifted interpolation functions have explicit evaluations. This is a sort of dual Karlsson-Minton sum; in particular, the dual of this sum (coming from the Corollary) is a multivariate analogue of [19, Cor. 4.5].

As usual with such sums, there is an integral analogue of (2.33). This was stated as a conjecture in the original version of this paper, and has since been proved by Van de Bult [3]. It has also appeared in a physical context in [24, §7].

Theorem 2.14. *[3, Thm. 3.1] For integers $m, n, n_0, \dots, n_{k-1} \geq 0$, and parameters t_0, t_1, t_2, t_3, v_0,…,v_{2k-1} satisfying*

$$t_0t_1t_2t_3 = t^{2+m-n} \tag{2.34}$$

$$v_{2i}v_{2i+1} = pq/t^{n_i} \tag{2.35}$$

$$\sum_{0\leq i<k} n_i = m+n, \tag{2.36}$$

one has

$$\begin{aligned} I\!I_n^{(k-1)}(t_0,t_1,t_2,t_3,v_0,\dots,v_{2k-1};t;p,q) = \prod_{m<i\leq n}\prod_{0\leq r<s<4}\Gamma_{p,q}(t^{n-i}t_rt_s)\prod_{0\leq i<2k}\prod_{0\leq r<4}\frac{\Gamma^+_{p,q,t}(pqt_r/v_i)}{\Gamma^+_{p,q,t}(t_rv_i)} \\ \times I\!I_m^{(k-1)}(t/t_0,t/t_1,t/t_2,t/t_3,v_0,\dots,v_{2k-1};t;p,q). \end{aligned} \tag{2.37}$$

Remark. Independently of [3], one can see that this holds when $k=1$ (both sides can be explicitly evaluated), as well as when $k=2$, as a special case of the E_8 symmetry of [15] (a rare case in which a transformation outside the usual double cosets can be applied, via a sequence of two dimension-changing transformations). The case $t\mapsto pq/t$, $|m-n|\leq 1$ appears naturally if one attempts to give a direct proof of the commutation relations for the integral operators of [15]. (Note that the case $m=n$ implies the general case, as one may take the limit $v_0\to t^{-n_0}t_3$ to reduce the dimension on the right-hand side.)

We will have occasion below to use the corresponding identity for commutation of difference operators.

Lemma 2.15. *For any parameters v_r such that $v_0v_1v_2v_3 = p^2q^2$, the BC_n-symmetric function*

$$\sum_{\sigma\in\{\pm1\}^n}\prod_{1\leq i\leq n}\frac{\prod_{0\leq r<4}\theta_p(v_rz_i^{\sigma_i})}{\theta_p(z_i^{2\sigma_i},pqz_i^{2\sigma_i})}\prod_{1\leq i<j\leq n}\frac{\theta_p(tz_i^{\sigma_i}z_j^{\sigma_j},(pq/t)z_i^{\sigma_i}z_j^{\sigma_j})}{\theta_p(z_i^{\sigma_i}z_j^{\sigma_j},pqz_i^{\sigma_i}z_j^{\sigma_j})} \tag{2.38}$$

is invariant under $v_r\mapsto pq/v_r$. In particular, the function

$$\sum_{\sigma\in\{\pm1\}^n}\prod_{1\leq i\leq n}\frac{\theta_p(qp^{1/2}w^{\pm1}z_i^{\sigma_i})}{\theta_p(p^{1/2}w^{\pm1}z_i^{\sigma_i})\theta_p(z_i^{2\sigma_i},pqz_i^{2\sigma_i})}\prod_{1\leq i<j\leq n}\frac{\theta_p(tz_i^{\sigma_i}z_j^{\sigma_j},(pq/t)z_i^{\sigma_i}z_j^{\sigma_j})}{\theta_p(z_i^{\sigma_i}z_j^{\sigma_j},pqz_i^{\sigma_i}z_j^{\sigma_j})} \tag{2.39}$$

is independent of w.

Proof. As shown in [15, 14], the composed difference operator

$$\mathcal{D}_q^{(n)}(u_0,t_0,t_1;t,p)\mathcal{D}_q^{(n)}(q^{1/2}u_0,q^{1/2}t_0,q^{-1/2}t_2;t,p) \tag{2.40}$$

is invariant under swapping t_1 and t_2, where

$$\begin{aligned} &(\mathcal{D}_q^{(n)}(a,b,c;t,p)f)(z_1,\dots,z_n) \\ &:= \sum_{\sigma\in\{\pm1\}^n}\prod_{1\leq i\leq n}\frac{\theta_p(az_i^{\sigma_i},bz_i^{\sigma_i},cz_i^{\sigma_i},t^{n-1}abcz_i^{-\sigma_i})}{\theta_p(z_i^{2\sigma_i},t^{n-i}ab,t^{n-i}ac,t^{n-i}cb)}\prod_{1\leq i<j\leq n}\frac{\theta_p(tz_i^{\sigma_i}z_j^{\sigma_j})}{\theta_p(z_i^{\sigma_i}z_j^{\sigma_j})}f(\dots,q^{\sigma_i/2}z_i,\dots). \end{aligned} \tag{2.41}$$

In particular, if we apply the composed operator to a function f, we obtain a linear combination of shifts of f, and each coefficient must be symmetric in t_1, t_2. Taking the coefficient of the unshifted term gives $\prod_{1\leq i\leq n}\theta_p(u_0z_i^{\pm1},t_0z_i^{\pm1})$ times the (general) instance $(v_0,v_1,v_2,v_3) = (t_1,pq/t_2,qt^{n-1}u_0t_0t_2,p/t^{n-1}u_0t_0t_1)$ of the above sum. □

Remark. More generally, if one takes the coefficient of some shift of f in which only m variables remain unshifted, then one obtains the $n = m$ instance of this sum, apart from some common factors. This gives a proof of this transformation and commutation of the difference operators without reference to the theory of interpolation functions: by induction on n, it follows that

$$\mathcal{D}_q^{(n)}(u_0,t_0,t_1;t,p)\mathcal{D}_q^{(n)}(q^{1/2}u_0,q^{1/2}t_0,q^{-1/2}t_2;t,p) - \mathcal{D}_q^{(n)}(u_0,t_0,t_2;t,p)\mathcal{D}_q^{(n)}(q^{1/2}u_0,q^{1/2}t_0,q^{-1/2}t_1;t,p) \quad (2.42)$$

acts as a scalar; to show that this scalar vanishes, one need simply apply it to 1, using the fact [15, Lem. 6.2] that

$$\mathcal{D}_q^{(n)}(a,b,c;t,p)1 = 1. \quad (2.43)$$

3 Elliptic Cauchy identities

From the results of the previous section, it is clear that the skew interpolation functions behave very much as analogues of skew Macdonald polynomials. This is not entirely surprising, given that skew Macdonald polynomials are limits of skew interpolation functions, as follows from Theorem 8.5 of [14]. More precisely, one has

$$\begin{aligned}\lim_{p\to 0} p^{|\lambda|/4-|\mu|/4}\,&\mathcal{R}^*_{(0,\lambda)/(0,\mu)}([p^{1/4}/v_0,\dots,p^{1/4}/v_{n-1},p^{-1/4}w_0,\dots,p^{-1/4}w_{n-1}];a,p^{1/2}b;t;p,q)\\ &= \frac{(-a)^{|\lambda|}q^{n(\lambda')}t^{-2n(\lambda)}C^-_\lambda(t;q,t)}{(-aV/W)^{|\mu|}q^{n(\mu')}t^{-2n(\mu)}C^-_\mu(t;q,t)}P_{\lambda/\mu}([\frac{v_0^k+\cdots+v_{n-1}^k-w_0^k-\cdots-w_{n-1}^k}{1-t^k}];q,t), \quad (3.1)\end{aligned}$$

by a straightforward induction from the case $n = 1$, when it reduces to [14, Thm. 8.5]. (Here

$$C^-_\lambda(x;q,t) := \mathcal{C}^-_{0,\lambda}(x;t;0,q) = \prod_{(i,j)\in\lambda}(1-q^{\lambda_i-j}t^{\lambda'_j-i}x) \quad (3.2)$$

is the usual hook-product symbol that appears in Macdonald polynomial theory (e.g., in the denominator of [10, (VI.6.11')], or in both numerator and denominator in [10, (VI.6.19)]), and the argument to $P_{\lambda/\mu}$ denotes the image under a homomorphism taking p_k to the stated value. As above, V denotes the product $v_0\cdots v_{n-1}$, and similarly for W.)

However, if we attempt to give a direct analogue of the Cauchy identity for skew Macdonald polynomials, we encounter the difficulty that sums of infinitely many elliptic terms rarely converge. It will thus be important to understand under what circumstances a skew interpolation function is forced to vanish.

Lemma 3.1. *If $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$ are partition pairs, l,m,n nonnegative integers, and a, b, and $v_0\in\mathbb{C}^*$ are generic, then*

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0,t^n/p^lq^mv_0];a,b;t;p,q) \quad (3.3)$$

vanishes unless

$$\boldsymbol{\kappa}_i \le \boldsymbol{\lambda}_i \le \boldsymbol{\kappa}_{i-n}+(l,m) \quad (3.4)$$

for all i, with the convention $\boldsymbol{\kappa}_0=\boldsymbol{\kappa}_{-1}=\cdots=\infty$.

Proof. Observe that we can write

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0, t^n/p^l q^m v_0]; a, b; t; p, q) = \sum_{\boldsymbol{\mu}} \mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([v_0, t^n/v_0]; a, b; t; p, q)\, \mathcal{R}^*_{\boldsymbol{\mu}/\boldsymbol{\kappa}}([v_0/t^n, t^n/p^l q^m v_0]; a/t^n, b; t; p, q), \tag{3.5}$$

with

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([v_0, t^n/v_0]; a, b; t; p, q) = \frac{\Delta^0_{\boldsymbol{\lambda}}(a/b|a/v_0, av_0/t^n; t; p, q)}{\Delta^0_{\boldsymbol{\mu}}(a/bt^n|a/v_0, av_0/t^n; t; p, q)} \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[ab,t^n];t;p,q}, \tag{3.6}$$

and

$$\mathcal{R}^*_{\boldsymbol{\mu}/\boldsymbol{\kappa}}([v_0/t^n, t^n/p^l q^m v_0]; a/t^n, b; t; p, q) = \frac{\Delta^0_{\boldsymbol{\mu}}(a/bt^n|a/v_0, av_0 p^l q^m/t^{2n}; t; p, q)}{\Delta^0_{\boldsymbol{\kappa}}(ap^l q^m/bt^n|a/v_0, av_0 p^l q^m/t^{2n}; t; p, q)} \left\langle \begin{matrix} \boldsymbol{\mu} \\ \boldsymbol{\kappa} \end{matrix} \right\rangle_{[a/bt^n, p^{-l}q^{-m}];t;p,q}. \tag{3.7}$$

The binomial coefficients vanish unless [14, Cor. 4.5]

$$\boldsymbol{\mu}_i \le \boldsymbol{\lambda}_i \le \boldsymbol{\mu}_{i-n} \tag{3.8}$$

and [14, Cor. 4.2]

$$\boldsymbol{\kappa}_i \le \boldsymbol{\mu}_i \le \boldsymbol{\kappa}_i + (l, m), \tag{3.9}$$

and (by genericity), this vanishing cannot be cancelled by a pole of the remaining factors. □

The other significant source of vanishing is the following.

Lemma 3.2. *If $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$ are partition pairs, l, m, n are nonnegative integers, a, b, and $v_0 \in \mathbb{C}^*$ are generic, and $\boldsymbol{\kappa}_{n+1} \le (l, m)$, then*

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0, a/p^{-l}q^{-m}t^n]; a, b; t; p, q) \tag{3.10}$$

vanishes unless $\boldsymbol{\lambda}_{n+1} \le (l, m)$.

Proof. We have

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0, a/p^{-l}q^{-m}t^n]; a, b; t; p, q) = \frac{\Delta^0_{\boldsymbol{\lambda}}(a/b|p^{-l}q^{-m}t^n; t; p, q)}{\Delta^0_{\boldsymbol{\kappa}}(p^{-l}q^{-m}t^n/bv_0|p^{-l}q^{-m}t^n; t; p, q)} \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\kappa} \end{matrix} \right\rangle_{[a/b, av_0/p^{-l}q^{-m}t^n](v_0);t;p,q}. \tag{3.11}$$

The binomial coefficient factor is generic, so cannot contribute any poles, as are the factors coming from denominators of Δ^0. We are thus left with considering the ratio

$$\frac{\mathcal{C}^0_{\boldsymbol{\lambda}}(p^{-l}q^{-m}t^n; t; p, q)}{\mathcal{C}^0_{\boldsymbol{\kappa}}(p^{-l}q^{-m}t^n; t; p, q)}. \tag{3.12}$$

If $\boldsymbol{\kappa}_{n+1} \le (l, m)$, then the denominator is nonzero; the numerator vanishes unless $\boldsymbol{\lambda}_{n+1} \le (l, m)$. □

Both lemmas extend by induction to vanishing conditions on more general skew interpolation functions.

Theorem 3.3. *Suppose the sequence $v_0, v_1, \dots, v_{2k-1}$ can be ordered in such a way that $v_{2i}v_{2i+1} = t^{n_i}p^{-l_i}q^{-m_i}$ with $l_i, m_i, n_i \ge 0$, for $0 \le i < k$, and are otherwise generic. Then for any partition pair $\boldsymbol{\kappa}$,*

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0, v_1, \dots, v_{2k-1}]; a, b; t; p, q) = 0 \tag{3.13}$$

unless

$$\boldsymbol{\kappa}_i \le \boldsymbol{\lambda}_i \le \boldsymbol{\kappa}_{i-N} + (L, M), \tag{3.14}$$

where $L = \sum_i l_i$, $M = \sum_i m_i$, $N = \sum_i n_i$.

Proof. If $k = 1$ or $k = 0$, this follows from Lemma 3.1. In general, we have

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0, v_1, \dots, v_{2k-1}]; a, b; t; p, q) = \sum_{\boldsymbol{\mu}} \mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([v_0, v_1]; a, b; t; p, q)\, \mathcal{R}^*_{\boldsymbol{\mu}/\boldsymbol{\kappa}}([v_2, \dots, v_{2k-1}]; a/v_0v_1, b; t; p, q); \tag{3.15}$$

the term associated to $\boldsymbol{\mu}$ vanishes unless

$$\boldsymbol{\mu}_i \le \boldsymbol{\lambda}_i \le \boldsymbol{\mu}_{i-n_0} + (l_0, m_0) \tag{3.16}$$

and (by induction)

$$\boldsymbol{\kappa}_i \le \boldsymbol{\mu}_i \le \boldsymbol{\kappa}_{i-(N-n_0)} + (L - l_0, M - m_0). \tag{3.17}$$

The claim follows. □

Similarly, one has the following.

Theorem 3.4. *Let* $l_0, \dots, l_{k-1}$, $m_0, \dots, m_{k-1}$, $n_0, \dots, n_{k-1}$ *be sequences of nonnegative integers, and suppose the otherwise generic sequence* $v_0, \dots, v_{2k-1}$ *can be ordered in such a way that* $v_{2i}v_{2i+1} = t^{n_i} p^{-l_i} q^{-m_i}$ *for* $1 \le i < k$, *while*

$$a / \prod_{1 \le i < 2k} v_i = t^{n_0} p^{-l_0} q^{-m_0}. \tag{3.18}$$

If $\boldsymbol{\kappa}_{n_0+1} \le (l_0, m_0)$, *then*

$$\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\kappa}}([v_0, v_1, \dots, v_{2k-1}]; a, b; t; p, q) = 0 \tag{3.19}$$

unless $\boldsymbol{\lambda}_{N+1} \le (L, M)$.

When $\boldsymbol{\kappa} = 0$, the two vanishing conditions coincide, and both simply state that $\boldsymbol{\lambda}_{N+1} \le (L, M)$. This corresponds to the extra symmetry explained in Proposition 2.6 above. One also obtains an additional (albeit more delicate) source of vanishing in the $\boldsymbol{\kappa} = 0$ case.

Theorem 3.5. *Let* $l \le L$; $m \le M$; $n \le N$ *be nonnegative integers. Then the lifted interpolation function*

$$\mathcal{R}^*_{\boldsymbol{\lambda}/0}([p^L q^M a/t^N, v_1, \dots, v_{2k-1}]; a, pqt^n/p^l q^m a; t; p, q) \tag{3.20}$$

vanishes unless $\boldsymbol{\lambda}_{N+1} \le (L, M)$.

Proof. We have the expansion

$$\begin{aligned} &\mathcal{R}^*_{\boldsymbol{\lambda}/0}([p^L q^M a/t^N, v_1, \dots, v_{2n-1}]; a, pqt^n/p^l q^m a; t; p, q) \\ &\quad = \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[a/b, t^n/p^l q^m]; t; p, q} \Delta^0_{\boldsymbol{\mu}}(pq/b^2 | t^{N-n}/p^{L-l} q^{M-m}, pq/bv_1, \dots, pq/bv_{2n-1}, pq \prod_{0 \le r < 2n} v_r/ab; t; p, q), \end{aligned} \tag{3.21}$$

where $b = pqt^n/p^lq^ma$. The second factor vanishes unless

$$\boldsymbol{\mu}_{N-n+1} \le (L-l, M-m), \tag{3.22}$$

while the first factor vanishes unless

$$\boldsymbol{\mu}_i \le \boldsymbol{\lambda}_i \le \boldsymbol{\mu}_{i-n} + (l,m). \tag{3.23}$$

□

Since infinite sums of elliptic functions tend not to converge, we need to insist in the elliptic Cauchy identity that the sum terminate; i.e., involve only finitely many terms. To avoid potential obstructions to analytic continuation arguments, we insist that the termination occurs either because the partition pair being summed over occurs as the lower partition in a skew interpolation function (or elliptic binomial coefficient), or because using either Δ^0 factors of the summand or one of the first two vanishing theorems (Theorem 3.3 or 3.4), one can bound both the first part and the length of the partition pair. In the latter case, we will refer to the source of the bound on the first part as a horizontal termination condition (as it bounds the horizontal extent of the corresponding diagram); similarly a vertical termination condition is one that allows us to bound the length. Note that a sum over skew diagrams which are unions of finitely many horizontal strips is vertically terminated, and vice versa.

With this in mind, we can now state our first version of the Cauchy identity for skew interpolation functions. Note that the termination conditions allow the right-hand side to be simplified to an expression in p-theta and q-theta functions; this would not hold if the sum were finite by virtue of the third vanishing condition alone.

Theorem 3.6. *One has the identity*

$$\begin{aligned}\sum_{\boldsymbol{\mu}} \Delta_{\boldsymbol{\mu}}(a/b|;t;p,q)\,\mathcal{R}^*_{\boldsymbol{\mu}/0}([v_0,\dots,v_{2n-1}];a,b;t;p,q)\,\mathcal{R}^*_{\boldsymbol{\mu}/0}([w_0,\dots,w_{2m-1}];\sqrt{pqt}/b,\sqrt{pqt}/a;t;p,q)\\ = \prod_{\substack{0\le i<2n+2\\0\le j<2m+2}} \frac{\Gamma^+_{p,q,t}((pqt)^{1/2}v_i/w_j)}{\Gamma^+_{p,q,t}((pqt)^{1/2}v_iw_j)},\end{aligned} \tag{3.24}$$

where

$$v_{2n} = a/\prod_{0\le r<2n} v_r, \quad v_{2n+1} = 1/a, \quad w_{2m} = (pqt)^{1/2}/b\prod_{0\le r<2m} w_r, \quad w_{2m+1} = b/(pqt)^{1/2}, \tag{3.25}$$

and the parameters are such that the sum terminates, but otherwise generic.

Proof. Suppose first that the vertical termination of the sum is due to the v parameters (i.e., the first skew interpolation function satisfies the hypotheses of Theorem 3.3 with $L = M = 0$ (or Theorem 3.4, but this is essentially equivalent in the case of lifted interpolation functions)), while the horizontal termination is due to the w parameters (the second skew interpolation function satisfies the hypotheses of Theorem 3.3 with $N = 0$). We may thus assume (adding or removing pairs $x, 1/x$ as necessary) that $v_{2i}v_{2i+1} = t$, $0 \le i < n$, while $w_{2i}w_{2i+1} = 1/p$ or $1/q$ for each $0 \le i < m$. In that case, we may factor the sum into the product of a q-elliptic sum and a p-elliptic sum. Applying duality to the w factor allows us to express both factors as interpolation functions, and the claim becomes the Cauchy identity of [14, Thm. 4.18].

The other possibility (up to obvious symmetries) is that one set of parameters (say the w parameters) provides both termination conditions. If the v parameters *also* provide vertical termination, then the result follows; in general, the set of v parameters for which $v_{2i}v_{2i+1} \in t^{\mathbb{N}}$, $0 \le i < n$, is Zariski dense on both elliptic curves, so we may analytically continue. $\square$

There is also a skew version of the above identity.

Theorem 3.7. *One has the identity*

$$\begin{aligned}\sum_{\boldsymbol{\mu}} \frac{\Delta_{\boldsymbol{\mu}}(a/b|;t;p,q)}{\Delta_{\boldsymbol{\lambda}}(a/bV|;t;p,q)} \mathcal{R}^*_{\boldsymbol{\mu}/\boldsymbol{\lambda}}([v_0,\dots,v_{2n-1}];a,b;t;p,q)\, \mathcal{R}^*_{\boldsymbol{\mu}/\boldsymbol{\kappa}}([w_0,\dots,w_{2m-1}];\sqrt{pqt}/b,\sqrt{pqt}/a;t;p,q)\\ \propto \sum_{\boldsymbol{\mu}} \mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}}([w_0,\dots,w_{2m-1}];\sqrt{pqt}/b,\sqrt{pqt}V/a;t;p,q) \frac{\Delta_{\boldsymbol{\kappa}}(a/bW|;t;p,q)}{\Delta_{\boldsymbol{\mu}}(a/bVW|;t;p,q)} \mathcal{R}^*_{\boldsymbol{\kappa}/\boldsymbol{\mu}}([v_0,\dots,v_{2n-1}];a,bW;t;p,q),\end{aligned} \tag{3.26}$$

where $V = \prod_r v_r$, $W = \prod_r w_r$, for generic parameters such that the left-hand side terminates. The constant of proportionality is independent of $\boldsymbol{\lambda}$ and $\boldsymbol{\kappa}$, and is thus equal to the value of the sum when $\boldsymbol{\lambda} = \boldsymbol{\kappa} = 0$.

Proof. First consider the case $\boldsymbol{\kappa} = 0$, so that only the term with $\boldsymbol{\mu} = 0$ survives on the right-hand side, and suppose furthermore that $v_{2i-1}v_{2i} = t$, and $w_{2i-1}w_{2i} \in p^{-\mathbb{N}}q^{-\mathbb{N}}$ for each i. If we multiply both sides by

$$\Delta_{\boldsymbol{\lambda}}(a/bV|;t;p,q)\, \mathcal{R}^*_{\boldsymbol{\lambda}/0}([t^{1/2}u_1^{\pm 1},\dots,t^{1/2}u_{\ell(\boldsymbol{\lambda})}^{\pm 1}];a/V,b;t;p,q) \tag{3.27}$$

and sum over $\boldsymbol{\lambda}$, the right-hand side becomes an instance of the previous theorem, while the left-hand side simplifies directly to an instance of the previous theorem. In particular, after so multiplying and summing, the two sides agree. But the test functions we have multiplied by are linearly independent, and thus both sides agree before summing.

The arbitrary terminating case with $\boldsymbol{\kappa} = 0$ then follows by analytic continuation. Similarly, the case $\boldsymbol{\kappa} \ne 0$ follows from the case $\boldsymbol{\kappa} = 0$, and the general claim follows by analytic continuation. $\square$

Another approach to proving the above identity is by induction on n and m; it suffices to consider the case $n = m = 1$, or in other words the following special case.

Corollary 3.8. *One has the identity*

$$\begin{aligned}\sum_{\boldsymbol{\mu}} \frac{\Delta_{\boldsymbol{\mu}}(a|v_0,v_1,v_2,v_3;t;p,q)}{\Delta_{\boldsymbol{\lambda}}(a/b_0|v_0,v_1,v_2,v_3;t;p,q)} \left\langle {\boldsymbol{\mu} \atop \boldsymbol{\lambda}} \right\rangle_{[a,b_0];t;p,q} \left\langle {\boldsymbol{\mu} \atop \boldsymbol{\kappa}} \right\rangle_{[a,b_1];t;p,q}\\ \propto \sum_{\boldsymbol{\mu}} \left\langle {\boldsymbol{\lambda} \atop \boldsymbol{\mu}} \right\rangle_{[a/b_0,b_1];t;p,q} \frac{\Delta_{\boldsymbol{\kappa}}(a/b_1|v_0,v_1,v_2,v_3;t;p,q)}{\Delta_{\boldsymbol{\mu}}(a/b_0b_1|v_0,v_1,v_2,v_3;t;p,q)} \left\langle {\boldsymbol{\kappa} \atop \boldsymbol{\mu}} \right\rangle_{[a/b_1,b_0];t;p,q},\end{aligned} \tag{3.28}$$

assuming the termination conditions

$$t^{\mathbb{N}} \cap \{b_0,b_1,v_0,v_1,v_2,v_3\} \ne \emptyset \tag{3.29}$$

$$p^{-\mathbb{N}}q^{-\mathbb{N}} \cap \{b_0,b_1,v_0,v_1,v_2,v_3\} \ne \emptyset, \tag{3.30}$$

(with corresponding conditions on $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$ if the v_r are used for termination), and the balancing condition $b_0b_1v_0v_1v_2v_3 = pqta^2$. The constant of proportionality is given by

$$\sum_{\boldsymbol{\mu}} \Delta_{\boldsymbol{\mu}}(a|b_0, b_1, v_0, v_1, v_2, v_3; t; p, q). \tag{3.31}$$

This, in turn, gives an expression for the (discrete) inner product of two interpolation functions with respect to the density of [14, Thm. 5.8]; in particular, it is a limiting case of an integral identity [15, Thm. 9.4]. It can also be obtained (following ideas of [18]) by computing connection coefficients for interpolation theta functions in two different ways (compare [14, Thm. 4.15]).

4 Elliptic Littlewood identities

Since the classical Littlewood identity only involves a single Schur function, the termination conditions in any direct elliptic analogue must be borne by a single skew interpolation function. It turns out, however, that the conditions can be weakened slightly; it is permissible for the skew interpolation function to allow unbounded upper partitions, so long as none of those satisfy the even multiplicity condition. The point is that since the first and second parts of $\boldsymbol{\mu}^2$ agree, we need only have a bound on the *second* part of $\boldsymbol{\mu}^2$ to obtain a terminating sum. Thus in the horizontal termination condition, we may allow one of the pairs to multiply to $tp^{-l_i}q^{-m_i}$ instead of $p^{-l_i}q^{-m_i}$.

In particular, if $v_0v_1 = t$, then this simultaneously gives both horizontal and vertical termination conditions. Indeed, we find that if $\mathcal{R}^*_{\boldsymbol{\mu}^2/\boldsymbol{\lambda}}([v_0, t/v_0]; a, b; t; p, q) \neq 0$, then

$$\begin{aligned} \boldsymbol{\lambda}_{2i-1} \leq &(\boldsymbol{\mu}^2)_{2i-1} \leq \boldsymbol{\lambda}_{2i-2} \qquad (4.1)\\ \boldsymbol{\lambda}_{2i} \leq &(\boldsymbol{\mu}^2)_{2i} \leq \boldsymbol{\lambda}_{2i-1}, \qquad (4.2) \end{aligned}$$

and thus, since $(\boldsymbol{\mu}^2)_{2i-1} = (\boldsymbol{\mu}^2)_{2i} = \boldsymbol{\mu}_i$,

$$\boldsymbol{\lambda}_{2i-1} \leq \boldsymbol{\mu}_i \leq \boldsymbol{\lambda}_{2i-1}, \tag{4.3}$$

so that $\boldsymbol{\mu}$ is uniquely determined. With this in mind, define new operations on partition pairs

$$\begin{aligned} \boldsymbol{\lambda}^+ : \boldsymbol{\lambda}^+_i &= \boldsymbol{\lambda}_{2i-1} \qquad (4.4)\\ \boldsymbol{\lambda}^- : \boldsymbol{\lambda}^-_i &= \boldsymbol{\lambda}_{2i-2}. \qquad (4.5) \end{aligned}$$

Lemma 4.1. *For any partition pair $\boldsymbol{\lambda}$,*

$$\begin{aligned} &\sum_{\boldsymbol{\mu}} \left\langle \begin{matrix} \boldsymbol{\mu}^2 \\ \boldsymbol{\lambda} \end{matrix} \right\rangle_{[a,t];t;p,q} \Delta_{\boldsymbol{\mu}}(a/t|v_0, \dots, v_{2n-1}; t^2; p, q)\\ &\qquad = \Delta_{\boldsymbol{\lambda}}(a/t|v_0, \dots, v_{2n-1}; t; p, q) \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu}^2 \end{matrix} \right\rangle_{[a/t,t];t;p,q} \frac{\Delta_{\boldsymbol{\mu}}(a/t^3|v_0, \dots, v_{2n-1}; t^2; p, q)}{\Delta_{\boldsymbol{\mu}^2}(a/t^2|v_0, \dots, v_{2n-1}; t; p, q)} \qquad (4.6) \end{aligned}$$

Proof. On the left-hand side, only the term with $\boldsymbol{\mu} = \boldsymbol{\lambda}^+$ contributes, while on the right-hand side, only the term with $\boldsymbol{\mu} = \boldsymbol{\lambda}^-$ contributes. Now, it follows easily from the definition of Δ^0 that

$$\Delta^0_{\boldsymbol{\lambda}}(a|v_0;t;p,q) = \Delta^0_{\boldsymbol{\lambda}^+}(a|v_0;t^2;p,q)\Delta^0_{\boldsymbol{\lambda}^-}(a/t^2|v_0/t;t^2;p,q), \tag{4.7}$$

$$\Delta^0_{\boldsymbol{\mu}^2}(a|v_0;t;p,q) = \Delta^0_{\boldsymbol{\mu}}(a/t|v_0,v_0/t;t^2;p,q), \tag{4.8}$$

and thus the dependence on v_r disappears. It thus suffices to show that

$$\left\langle \begin{matrix} \boldsymbol{\lambda}^{+2} \\ \boldsymbol{\lambda} \end{matrix} \right\rangle_{[a,t];t;p,q} = \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\lambda}^{-2} \end{matrix} \right\rangle_{[a/t,t];t;p,q} \frac{\Delta_{\boldsymbol{\lambda}}(a/t|;t;p,q)\Delta_{\boldsymbol{\lambda}^-}(a/t^3|;t^2;p,q)}{\Delta_{\boldsymbol{\lambda}^{-2}}(a/t^2|;t;p,q)\Delta_{\boldsymbol{\lambda}^+}(a/t|;t^2;p,q)}. \tag{4.9}$$

This can be proved by induction on $\ell(\boldsymbol{\lambda})$ via the observations

$$((l,m)\cdot\boldsymbol{\lambda})^+ = (l,m)\cdot\boldsymbol{\lambda}^- \tag{4.10}$$

$$((l,m)\cdot\boldsymbol{\lambda})^- = \boldsymbol{\lambda}^+ \tag{4.11}$$

and the relation [14, Cor. 4.8]

$$\left\langle \begin{matrix} (l,m)\cdot\boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[a,t];t;p,q} = \Delta^0_{(l,m)}(a|t;t;p,q)\frac{\Delta_{\boldsymbol{\mu}}(a/t|p^{-l}q^{-m},p^lq^ma;t;p,q)}{\Delta_{\boldsymbol{\lambda}}(a/t^2|p^{-l}q^{-m},p^lq^ma;t;p,q)}\left\langle \begin{matrix} \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{matrix} \right\rangle_{[a/t,t];t;p,q}. \tag{4.12}$$

Note that we may freely check the p-theta and q-theta portions of the relation separately, and rescale so that both are elliptic. $\square$

The first version of an elliptic Littlewood identity is the following.

Theorem 4.2. *We have*

$$\sum_{\boldsymbol{\mu}} \mathcal{R}^*_{\boldsymbol{\mu}^2/0}([v_0,\dots,v_{2n-1}];ta,(pqt)^{1/2}/a;t;p,q)\Delta_{\boldsymbol{\mu}}(a^2/(pqt)^{1/2}|;t^2;p,q)$$
$$= \frac{\Gamma^+_{p,q,t}((pqt)^{1/2})^n\Gamma^+_{p,q,t}((pqt)^{1/2}t)\prod_{0\le i<j<2n+2}\Gamma^+_{p,q,t}((pqt)^{1/2}v_i/v_j)}{\prod_{0\le i<2n+2}\Gamma^+_{p,q,t^2}((pqt)^{1/2}v_i^2)\prod_{0\le i<j<2n+2}\Gamma^+_{p,q,t}((pqt)^{1/2}v_iv_j)}, \tag{4.13}$$

where $v_{2n} = ta/\prod_{0\le i<2n} v_i$, $v_{2n+1} = 1/a$, and the sum terminates.

Proof. Using the S_{2n+1} symmetry of the lifted interpolation functions, we may assume (inserting $x, 1/x$ pairs as necessary) that the parameters pairwise multiply to t, and are ordered in such a way that $v_{2m},\dots,v_{2n-1}$ gives both horizontal and vertical termination conditions for $0 \le m < n$. The proof then follows by a straightforward induction on n:

$$\sum_{\boldsymbol{\mu}} \mathcal{R}^*_{\boldsymbol{\mu}^2/0}([v_0,\dots,v_{2n-1}];ta,b;t;p,q)\Delta_{\boldsymbol{\mu}}(a/b|;t^2;p,q)$$
$$= \sum_{\boldsymbol{\lambda}} \mathcal{R}^*_{\boldsymbol{\lambda}/0}([v_2,\dots,v_{2n-1}];a,b;t;p,q)\sum_{\boldsymbol{\mu}} \mathcal{R}^*_{\boldsymbol{\mu}^2/\boldsymbol{\lambda}}([v_0,t/v_0];ta,b;t;p,q)\Delta_{\boldsymbol{\mu}}(a/b|;t^2;p,q) \tag{4.14}$$
$$= \sum_{\boldsymbol{\mu}} \frac{\Delta_{\boldsymbol{\mu}}(a/bt^2|;t^2;p,q)}{\Delta_{\boldsymbol{\mu}^2}(a/bt|;t;p,q)}$$
$$\times \sum_{\boldsymbol{\lambda}} \Delta_{\boldsymbol{\lambda}}(a/b|;t;p,q)\,\mathcal{R}^*_{\boldsymbol{\lambda}/0}([v_2,\dots,v_{2n-1}];a,b;t;p,q)\,\mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}^2}([v_0,t/v_0];a,b;t;p,q) \tag{4.15}$$
$$\propto \sum_{\boldsymbol{\mu}} \Delta_{\boldsymbol{\mu}}(a/bt^2|;t^2;p,q)\,\mathcal{R}^*_{\boldsymbol{\mu}^2/0}([v_2,\dots,v_{2n-1}];a,bt;t;p,q). \tag{4.16}$$

Note that the last step only works if $ab = (pqt)^{1/2}$. ☐

Remark. Note that the termination condition prevents one from obtaining a Macdonald polynomial identity as a simple limit, except in the case $n = 1$. However, if one ignores the issue of termination, and takes a limit above, one obtains

$$\begin{aligned}\sum_\mu \frac{C^-_\mu(t;q,t^2)}{C^-_\mu(q;q,t^2)} P_{\mu^2}([\frac{v_0^k + \cdots + v_{n-1}^k - w_0^k - \cdots - w_{n-1}^k}{1-t^k}];q,t)& \\ = \frac{\prod_{0\le i,j<n}(v_i w_j;q,t)}{\prod_{0\le i<n}(w_i^2, tv_i^2;q,t^2)\prod_{0\le i<j<n}(v_i v_j, w_i w_j;q,t)},&\end{aligned} \tag{4.17}$$

agreeing with Macdonald's q,t-Littlewood identity. This agreement results from the fact that the $n = 1$ case and the Cauchy identity together suffice to make the above induction work in the absence of termination.

If the lifted interpolation function is terminating in the usual sense (i.e., without taking advantage of the one extra factor of t), then it in fact corresponds to an ordinary interpolation function evaluated at a partition. This gives rise to the following curious identity.

Corollary 4.3. *For every partition pair $\boldsymbol{\lambda}$, one has the following identity of meromorphic functions*

$$\sum_{\boldsymbol{\mu}} \frac{\Delta_{\boldsymbol{\mu}}(a/(pqt)^{1/2}t|;t^2;p,q)}{\Delta_{\boldsymbol{\mu}^2}(a/(pqt)^{1/2}|;t;p,q)} \left\langle \begin{matrix}\boldsymbol{\lambda} \\ \boldsymbol{\mu}^2\end{matrix} \right\rangle_{[a,(pqt)^{1/2}];t;p,q} = \frac{\mathcal{C}^-_{\boldsymbol{\lambda}}((pqt)^{1/2};t;p,q)\,\mathcal{C}^+_{\boldsymbol{\lambda}}((pqt)^{1/2}a/t;t;p,q)}{\mathcal{C}^0_{2\boldsymbol{\lambda}}((pqt)^{1/2}a/t;t^2;p,q)}. \tag{4.18}$$

Following the argument of Theorem 3.7, one has the following skew analogue of the Littlewood identity.

Theorem 4.4. *The following identity holds:*

$$\begin{aligned}\sum_{\boldsymbol{\mu}} \mathcal{R}^*_{\boldsymbol{\mu}^2/\boldsymbol{\lambda}}([v_0,\dots,v_{2n-1}];ta,(pqt)^{1/2}/a;t;p,q)\frac{\Delta_{\boldsymbol{\mu}}(a^2/(pqt)^{1/2}|;t^2;p,q)}{\Delta_{\boldsymbol{\lambda}}(a^2t/(pqt)^{1/2}V|;t;p,q)}& \\ \propto \sum_{\boldsymbol{\mu}} \mathcal{R}^*_{\boldsymbol{\lambda}/\boldsymbol{\mu}^2}([v_0,\dots,v_{2n-1}];a,(pqt)^{1/2}V/ta;t;p,q)\frac{\Delta_{\boldsymbol{\mu}}(a^2/(pqt)^{1/2}V^2|;t^2;p,q)}{\Delta_{\boldsymbol{\mu}^2}(a^2t/(pqt)^{1/2}V^2|;t;p,q)},&\end{aligned} \tag{4.19}$$

assuming the LHS terminates; the constant of proportionality is independent of $\boldsymbol{\lambda}$, and can be obtained by setting $\boldsymbol{\lambda} = 0$.

Proof. One can again proceed by induction on n; for $n > 1$, a terminating case always has a pair multiplying to t (possibly after adding a pair multiplying to 1) such that the various sums continue to terminate after extracting that pair. One thus reduces to the case $n = 1$; if $v_0v_1 = t$, this has already been shown, while in general it follows from Theorem 4.5 below. ☐

Remark. Again, this formally produces Macdonald's skew q,t-Littlewood identity in the limit.

One disappointing aspect of the above identities is the fact that ab (or, in the case of binomial coefficients, b) is constrained. It appears that this is a necessary constraint if we wish a completely general Littlewood identity, but if we are willing to restrict our attention to binomial coefficients, we can introduce more parameters.

Theorem 4.5. *If $b^2v_0v_1v_2v_3 = pqta^2$, and the LHS terminates, then*

$$\sum_{\boldsymbol{\mu}} \left\langle \begin{matrix} \boldsymbol{\mu}^2 \\ \boldsymbol{\lambda} \end{matrix} \right\rangle_{[a,b];t;p,q} \frac{\Delta_{\boldsymbol{\mu}}(a/t|v_0,v_1,v_2,v_3;t^2;p,q)}{\Delta_{\boldsymbol{\lambda}}(a/b|v_0,v_1,v_2,v_3;t;p,q)} \propto \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu}^2 \end{matrix} \right\rangle_{[a/b,b];t;p,q} \frac{\Delta_{\boldsymbol{\mu}}(a/tb^2|v_0,v_1,v_2,v_3;t^2;p,q)}{\Delta_{\boldsymbol{\mu}^2}(a/b^2|v_0,v_1,v_2,v_3;t;p,q)} \tag{4.20}$$

where the constant of proportionality is independent of $\boldsymbol{\lambda}$. The termination condition on the LHS is that

$$t^{2\mathbb{N}} \cap \{v_0,v_1,v_2,v_3,b,b/t\} \neq \emptyset \tag{4.21}$$

$$p^{-\mathbb{N}}q^{-\mathbb{N}} \cap \{v_0,v_1,v_2,v_3,b,b/t\} \neq \emptyset, \tag{4.22}$$

with corresponding constraints on $\boldsymbol{\lambda}$ if a v_r is used for termination.

Proof. If we write

$$\begin{aligned}\Delta^0_{\boldsymbol{\mu}}(a/t|v_3;t^2;p,q)\left\langle \begin{matrix} \boldsymbol{\mu}^2 \\ \boldsymbol{\lambda} \end{matrix} \right\rangle_{[a,b];t;p,q} &= \Delta^0_{\boldsymbol{\mu}}(a/t|bv_3/t,bv_3/t^2,pqa/v_3;t^2;p,q) \\ &\quad\times \sum_{\boldsymbol{\nu}} \left\langle \begin{matrix} \boldsymbol{\mu}^2 \\ \boldsymbol{\nu} \end{matrix} \right\rangle_{[a,t];t;p,q} \left\langle \begin{matrix} \boldsymbol{\nu} \\ \boldsymbol{\lambda} \end{matrix} \right\rangle_{[a/t,b/t](v_3/t,pqa/bv_3);t;p,q}\end{aligned} \tag{4.23}$$

(where v_3 is not used for termination), and apply Lemma 4.1, we reduce to the case with

$$(a,b,v_3) \mapsto (a/t^2,b/t,v_3/t); \tag{4.24}$$

thus by induction (the claim being trivial when $b=1$), we obtain every case with $b \in t^{\mathbb{N}}$, and the general result by analytic continuation. ☐

Remark. When $\boldsymbol{\lambda} = 0$, the right-hand sum becomes 1, while the left-hand side becomes

$$\sum_{\boldsymbol{\mu}} \Delta_{\boldsymbol{\mu}}(a/t|v_0,v_1,v_2,v_3,b,b/t;t^2;p,q), \tag{4.25}$$

which can be evaluated, thus determining the normalization.

Corollary 4.6. *If $v_0v_1v_2v_3 = p^{2l+2}q^{2m+2}a^2$ and $\ell(\boldsymbol{\lambda}) \leq 2n$ with $l,m,n \in \mathbb{N}$, then*

$$\sum_{\boldsymbol{\mu}} \left\langle \begin{matrix} \boldsymbol{\mu}^2 \\ \boldsymbol{\lambda} \end{matrix} \right\rangle_{[a,p^{-l}q^{-m}];t;p,q} \Delta_{\boldsymbol{\mu}}(a/t|t^{2n},a/t^{2n};t^2;p,q) \prod_{0\leq r<4} \frac{\Delta^0_{\boldsymbol{\mu}}(a/t|v_r;t^2;p,q)}{\mathcal{C}^0_{(l,m)^n}(pqt^{2n-1}/v_r;t^2;p,q)\,\mathcal{C}^0_{\boldsymbol{\lambda}}(v_r;t;p,q)} \tag{4.26}$$

is invariant under $v_r \mapsto p^{l+1}q^{m+1}a/v_r$.

Proof. When $v_0 = pqt^{2n-1}$ and $\boldsymbol{\lambda}_{2n} \geq (l,m)$, the left-hand side simplifies to the case $v_0 = t^{2n}$, $b = p^{-l}q^{-m}$ of the left-hand side of Theorem 4.5, while the right-hand sides become equivalent after applying Corollary 4.8 of [14]. Substituting $\boldsymbol{\lambda} \mapsto (l',m')^{2n} + \boldsymbol{\lambda}$ for $l' \geq l$, $m' \geq m$, then shifting the variable of summation gives the case $v_0 = p^{l'+1}q^{m'+1}t^{2n-1}$ of the corollary. Since these cases are Zariski dense, the corollary follows. ☐

We observed above that Corollary 3.8 can be interpreted as giving the inner product of two interpolation functions, and is in particular a special case of a more general integral identity. The same applies to Theorem 4.5. The basic observation is that the sequence of points

$$t^{2n-i}(p,q)^{(\boldsymbol{\mu}^2)_i}a, \quad 1 \leq i \leq 2, \tag{4.27}$$

which arises when evaluating an interpolation function at $\boldsymbol{\mu}^2$, can also be expressed in the form

$$t^{\pm 1/2}(t^2)^{n-i}(p,q)^{\boldsymbol{\mu}_i}t^{1/2}a, \quad 1\le i\le n. \tag{4.28}$$

This gives rise to the following result, where we recall that

$$\langle f(z_1,\dots,z_n)\rangle^{(n)}_{t_0,t_1,t_2,t_3,u_0,u_1;t;p,q} \tag{4.29}$$

is the normalized linear functional associated to the n-dimensional elliptic Selberg integral.

Theorem 4.7. *For any partition pair $\boldsymbol{\lambda}$, and generic parameters satisfying the balancing condition*

$$t^{4n-2}t_0t_1t_2t_3u_0^2 = pq, \tag{4.30}$$

one has

$$\begin{aligned}
&\langle \mathcal{R}^{*(2n)}_{\boldsymbol{\lambda}}(\dots,t^{\pm 1/2}z_i,\dots;t_0,u_0;t;p,q)\rangle^{(n)}_{t^{1/2}t_0,t^{1/2}t_1,t^{1/2}t_2,t^{1/2}t_3,t^{\pm 1/2}u_0;t^2;p,q}\\
&\quad= \Delta^0_{\boldsymbol{\lambda}}(t^{2n-1}t_0/u_0|t^{2n-1}t_0t_1,t^{2n-1}t_0t_2,t^{2n-1}t_0t_3;t;p,q)\\
&\qquad\times\sum_{\boldsymbol{\mu}}\left\langle{\boldsymbol{\lambda}\atop \boldsymbol{\mu}^2}\right\rangle_{[t^{2n-1}t_0/u_0,t^{2n-1}t_0u_0];t;p,q}\frac{\Delta_{\boldsymbol{\mu}}(1/tu_0^2|t^{2n},t^{2n-1}t_0t_1,t^{2n-1}t_0t_2,t^{2n-1}t_0t_3;t^2;p,q)}{\Delta_{\boldsymbol{\mu}^2}(1/u_0^2|t^{2n},t^{2n-1}t_0t_1,t^{2n-1}t_0t_2,t^{2n-1}t_0t_3;t;p,q)}
\end{aligned}\tag{4.31}$$

Proof. If $t^{2n-1}t_0t_1 = p^{-l}q^{-m}$, so that the integral reduces to a sum, the identity is a case of Theorem 4.5. But the left-hand side can be computed by expanding

$$\mathcal{R}^{*(2n)}_{\boldsymbol{\lambda}}(\dots,t^{\pm 1/2}z_i,\dots;t_0,u_0;t;p,q) \tag{4.32}$$

as a linear combination of products of a function

$$\tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\mu}}(\dots,z_i,\dots;t^{1/2}t_0{:}t^{1/2}t_1,t^{1/2}t_2,t^{1/2}t_3;t^{1/2}u_0,t^{-1/2}u_0;t^2;p,q) \tag{4.33}$$

and a function

$$\tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\nu}}(\dots,z_i,\dots;t^{1/2}t_0{:}t^{1/2}t_1,t^{1/2}t_2,t^{1/2}t_3;t^{-1/2}u_0,t^{1/2}u_0;t^2;p,q). \tag{4.34}$$

(This is not to say that this expansion can be done explicitly; it suffices that such an expansion exists, which follows from the fact that all allowed poles are covered.) In particular, it follows that the left-hand side is the product of p- and q-theta functions, as is the right-hand side, so we may analytically continue to obtain the desired result. ∎

The following special case has a particularly simple summand.

Corollary 4.8. *If $t^{4n-2}t_0^2u_0^2v_0v_1 = pq$, then*

$$\begin{aligned}
&\left\langle \frac{\mathcal{R}^{*(2n)}_{\boldsymbol{\lambda}}(\dots,t^{\pm 1/2}z_i,\dots;t^{1/2}t_0,t^{1/2}u_0;t;p,q)}{\Delta^0_{\boldsymbol{\lambda}}(t^{2n-1}t_0/u_0|t^{2n-1}t_0v_0,t^{2n-1}t_0v_1;t;p,q)}\right\rangle^{(n)}_{t_0,tt_0,u_0,tu_0,v_0,v_1;t^2;p,q}\\
&= \Delta^0_{\boldsymbol{\lambda}}(t^{2n-1}t_0/u_0|t^{2n-1}t_0^2;t;p,q)\sum_{\boldsymbol{\mu}}\left\langle{\boldsymbol{\lambda}\atop \boldsymbol{\mu}^2}\right\rangle_{[t^{2n-1}t_0/u_0,t^{2n}t_0u_0];t;p,q}\frac{\Delta_{\boldsymbol{\mu}}(1/t^2u_0^2|t^{2n},t^{2n-1}t_0^2;t^2;p,q)}{\Delta_{\boldsymbol{\mu}^2}(1/tu_0^2|t^{2n},t^{2n-1}t_0^2;t;p,q)}
\end{aligned}\tag{4.35}$$

If we multiply both sides of Theorem 4.7 by

$$\Delta^0_{\boldsymbol{\lambda}}(t^{2n-1}t_0/u_0|t^{2n-1}t_0t_1, t^{2n-1}t_0t_2, t^{2n-1}t_0t_3; t; p, q)^{-1} \left\langle \begin{matrix} \boldsymbol{\kappa} \\ \boldsymbol{\lambda} \end{matrix} \right\rangle_{[1/u_0^2, 1/t^{2n-1}t_0u_0]; t; p, q} \tag{4.36}$$

and sum over $\boldsymbol{\lambda}$, the right-hand sum collapses to a delta function, and thus vanishes unless $\boldsymbol{\kappa} = \boldsymbol{\mu}^2$ for some $\boldsymbol{\mu}$. The effect on the left-hand side is to produce a biorthogonal function, and we thus obtain the following vanishing identity.

Corollary 4.9. *For generic parameters such that $t^{4n-2}t_0t_1t_2t_3u_0^2 = pq$, the integral*

$$\langle \tilde{\mathcal{R}}^{(2n)}_{\boldsymbol{\lambda}}(\dots, t^{\pm 1/2}z_i, \dots; t_0{:}t_1, t_2, t_3; u_0, u_0; t; p, q)\rangle^{(n)}_{t^{1/2}t_0, t^{1/2}t_1, t^{1/2}t_2, t^{1/2}t_3, t^{\pm 1/2}u_0; t^2; p, q} \tag{4.37}$$

vanishes unless $\boldsymbol{\lambda} = \boldsymbol{\mu}^2$ for some partition pair $\boldsymbol{\mu}$, in which case it equals

$$\frac{\Delta_{\boldsymbol{\mu}}(1/tu_0^2|t^{2n}, t^{2n-1}t_0t_1, t^{2n-1}t_0t_2, t^{2n-1}t_0t_3, 1/t^{2n-1}t_0u_0, 1/t^{2n}t_0u_0; t^2; p, q)}{\Delta_{\boldsymbol{\mu}^2}(1/u_0^2|t^{2n}, t^{2n-1}t_0t_1, t^{2n-1}t_0t_2, t^{2n-1}t_0t_3, 1/t^{2n-1}t_0u_0, 1/t^{2n-1}t_0u_0; t; p, q)}. \tag{4.38}$$

Remark. In fact, although we have referred to the functions above as "biorthogonal" functions, since $u_0 = u_1$, they actually form an orthogonal basis of the appropriate space of functions.

If we fix t_0, t_1, t_2, t_3 and let $p \to 0$ (solving for u_0 via the balancing condition, so that $u_0 \sim \sqrt{p}$), the biorthogonal functions converge to Koornwinder polynomials, and the density converges to a (different) Koornwinder density. The result is one of the vanishing integrals of [17] (Theorem 4.9 op. cit.), *together* with the nonzero values (which were not accessible to the methods used there). In the notation of [13], one has

$$\begin{aligned} &I_K(\tilde{K}_\lambda([p_k(t^{k/2} + t^{-k/2})]; q, t; T; t_0, t_1, t_2, t_3); q, t^2, T; t^{1/2}t_0, t^{1/2}t_1, t^{1/2}t_2, t^{1/2}t_3) \\ &= \delta_{\lambda\mu^2} \frac{t^{-|\mu|} \prod_{0 \le r < s \le 3} C^0_\mu(Tt_rt_s/t; q, t^2) C^0_\mu(T, Tt_0t_1t_2t_3/t^2; q, t^2) C^+_\mu(T^2t_0t_1t_2t_3/t^4; q, t^2) C^-_\mu(qt; q, t^2)}{C^0_{2\mu^2}(T^2t_0t_1t_2t_3/t^2; q, t^2) C^+_\mu(T^2t_0t_1t_2t_3/qt^3; q, t^2) C^-_\mu(t^2; q, t^2)}. \end{aligned} \tag{4.39}$$

(If $T = t^{2n}$, this states that the integral of $K^{(2n)}_\lambda(\dots, t^{\pm 1/2}z_i, \dots; q, t; t_0, t_1, t_2, t_3)$ against the normalized density with parameters $q, t^2; t^{1/2}t_0, t^{1/2}t_1, t^{1/2}t_2, t^{1/2}t_3$ vanishes unless $\lambda = \mu^2$, when the value is as given.)

We furthermore conjecture that Theorem 4.7 extends to the following transformation (much as Corollary 3.8 extends to Theorem 9.7 of [15]). For the significance of the label $(t^{-1/2})$, see the end of Section 5.

Conjecture L1 $(t^{-1/2})$**.** *For generic parameters such that $t^{4n-4}t_0^2u_0^2v_0v_1v_2v_3 = p^2q^2$, one has*

$$\begin{aligned} &\int \mathcal{R}^{*(2n)}_{\boldsymbol{\lambda}}(\dots, t^{\pm 1/2}z_i, \dots; t_0, u_0; t; p, q)\Delta^{(n)}(; t^{\pm 1/2}t_0, t^{\pm 1/2}u_0, v_0, v_1, v_2, v_3; t^2; p, q) \\ &\quad = \prod_{0 \le r \le 3} \Delta^0_{\boldsymbol{\lambda}}(t^{2n-1}t_0/u_0|t^{2n-3/2}t_0v_r; t; p, q) \prod_{1 \le i \le 2n} \Gamma_{p,q}(t^{2n-1/2-i}t_0v_r, t^{2n-1/2-i}u_0v_r) \\ &\qquad \times \int \mathcal{R}^{*(2n)}_{\boldsymbol{\lambda}}(\dots, t^{\pm 1/2}z_i, \dots; t_0, u_0; t; p, q)\Delta^{(n)}(; t^{\pm 1/2}t_0, t^{\pm 1/2}u_0, v_0', v_1', v_2', v_3'; t^2; p, q), \end{aligned} \tag{4.40}$$

where $v_r' = pq/t^{2n-2}t_0u_0v_r$.

This is accessible in a number of special cases. When $t^{-1/2}t_0v_0 = pq$, so the left-hand side reduces to the left-hand side of Theorem 4.7, the transformed parameters satisfy $t^{2n-2}t^{1/2}u_0v_0' = 1$, and thus the right-hand side degenerates to a sum. If one traces through the relevant contour conditions, one finds that the sum is over partitions contained in $\boldsymbol{\mu}$, and one obtains the right-hand side of Theorem 4.7. It follows, then, (using the consistency under parameter shifts, below) that any "algebraic" case (i.e., in which both sides can be renormalized to products of p- and q-theta functions) of the conjecture holds.

When $\ell(\boldsymbol{\lambda}) = 1$, the skew interpolation function is independent of t, which implies

$$\mathcal{R}^{*(2n)}_{\boldsymbol{\lambda}}(\dots, t^{\pm 1/2} z_i, \dots; t_0, u_0; t; p, q) = \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots, z_i, \dots; t^{1/2} t_0, t^{-1/2} u_0; t^2; p, q) \tag{4.41}$$

when $\ell(\boldsymbol{\lambda}) \le 1$. Thus in that case, the conjecture becomes a special case of [15, Thm. 9.7]. We also find that the identity for $(l, m)^{2n} + \boldsymbol{\lambda}$ follows trivially from that for $\boldsymbol{\lambda}$; combining these two facts proves the identity when $n = 1$, and then trivially the case $t = 1$. In the case $t^{2n} t_0 u_0 = pq$, the interpolation function is of Cauchy type, and thus factors for general $\boldsymbol{\lambda}$:

$$\begin{aligned}\mathcal{R}^{*(2n)}_{\boldsymbol{\lambda}}(\dots, t^{\pm 1/2} z_i, \dots; t_0, pq/t^{2n} t_0; t; p, q) = {} & \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}^+}(\dots, z_i, \dots; t^{1/2} t_0, pq/t^{2n+1/2} t_0; t^2; p, q) \\ & \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}^-}(\dots, z_i, \dots; t^{-1/2} t_0, pq/t^{2n-1/2} t_0; t^2; p, q),\end{aligned} \tag{4.42}$$

and again the identity reduces to the transformation of [15, Thm. 9.7]. The case $t = q$ (and, by symmetry, $t = p$) can be dealt with via the observation that the p-elliptic interpolation functions can in that case be expressed as a ratio of determinants, while the q-elliptic interpolation function is a symmetrized product, just as for $t = 1$. More precisely, one has

$$\mathcal{R}^{*(n)}_{\lambda,\mu}(\dots, z_i, \dots; t_0, u_0; q; p, q) \propto \frac{\sum_{\pi,\rho \in S_n} \sigma(\rho) \prod_{1 \le i \le n} \mathcal{R}^{*(1)}_{\lambda_{\pi_i}, \mu_{\rho_i} + n - \rho_i}(z_i; t_0, q^{n-1} u_0; q; p, q)}{\prod_{1 \le i \le n} \theta_p(u_0 z_i^{\pm 1}; q)^{-1}_{n-1} \prod_{1 \le i < j \le n} z_i^{-1} \theta_p(z_i z_j^{\pm 1})}. \tag{4.43}$$

(Since the same interpolation function appears on both sides of (4.40), we can freely ignore constants.) After specializing the variables, the denominator cancels out the cross terms from the density, so that one can express the identity as a sum of products of instances with $n = 1$. (When $\boldsymbol{\lambda} = (0, \mu)$, this sum of products is a pfaffian, compare [2].)

One final set of special cases is of interest.

Proposition 4.10. *Conjecture L1 holds whenever $t^{2n-2} t_0 u_0 v_0 v_1 / pq \in \{1, 1/p, 1/q, t\}$.*

Proof. If $t^{2n-2} t_0 u_0 v_0 v_1 / pq = 1$, then the transformation is trivial. If $t^{2n-2} t_0 u_0 v_0 v_1 / pq = t$, we may use the integral equation [15, (8.12)] to write

$$\begin{aligned}\mathcal{R}^{*(2n)}_{\boldsymbol{\lambda}}(; t_0, u_0; t; p, q) = {} & \Delta^0_{\boldsymbol{\lambda}}(t^{2n-1} t_0 / u_0 | t^{2n-3/2} t_0 v_0, t^{2n-3/2} t_0 v_1; t; p, q) \\ & \times \mathcal{I}^{(2n)}(t^{-1/2} u_0 {:} t^{-1/2} t_0, t^{-1} v_0; p, q)\, \mathcal{R}^{*(2n)}_{\boldsymbol{\lambda}}(; t^{-1/2} t_0, t^{-1/2} u_0; t; p, q),\end{aligned} \tag{4.44}$$

and thus reduce to showing that

$$\begin{aligned}& \Gamma_{p,q}(t^{-3/2} t_0 v_0, t^{-3/2} u_0 v_0, t^{-3/2} t_0 v_1, t^{-3/2} u_0 v_1) \\ & \times \int (\mathcal{I}^{(2n)}(t^{-1/2} u_0 {:} t^{-1/2} t_0, v_0/t; p, q) f)(\dots, t^{\pm 1/2} z_i, \dots) \Delta^{(n)}(; t^{\pm 1/2} t_0, t^{\pm 1/2} u_0, v_0, v_1, v_2, v_3; t^2; p, q)\end{aligned} \tag{4.45}$$

is invariant under $(v_0, v_1, v_2, v_3) \mapsto (v_2', v_3', v_0', v_1')$ for any function f in the span of the interpolation functions. Now, specializing the output of the integral operator pinches the contour, and thus we pick up an n-fold residue. We thus find in general that if $t^{2n}u_0u_1u_2u_3 = pq$, then

$$\prod_{0\le r<s<4} \Gamma_{p,q}(u_r u_s)(\mathcal{I}_t^{(2n)}(u_0{:}u_1, u_2; p, q)f)(\dots, t^{\pm 1/2}z_i, \dots)$$
$$= \frac{((p;p)(q;q)/2\Gamma_{p,q}(t^2))^n}{n!} \int_{C^n} f(x_1, \dots, x_n, z_1, \dots, z_n) \frac{\prod_{1\le i,j\le n} \Gamma_{p,q}(t x_i^{\pm 1} z_j^{\pm 1})}{\prod_{1\le i<j\le n} \Gamma_{p,q}(x_i^{\pm 1} x_j^{\pm 1}, t^2 z_i^{\pm 1} z_j^{\pm 1})}$$
$$\times \prod_{1\le i\le n} \frac{\prod_{0\le r<4} \Gamma_{p,q}(u_r x_i^{\pm 1})}{\Gamma_{p,q}(x_i^{\pm 2}) \prod_{0\le r<4} \Gamma_{p,q}(t u_r z_i^{\pm 1})} \frac{dx_i}{2\pi\sqrt{-1}x_i}. \tag{4.46}$$

Substituting in and exchanging order of integration gives the desired result.

By symmetry, it remains only to consider the case $t^{2n-2}t_0u_0v_0v_1/pq = 1/q$. Here we use the difference equation [15, (8.11)] rather than the integral equation, and reduce to showing that

$$\int (D_q^{(2n)}(u_0, t_0, t^{-1/2}v_0, t^{-1/2}v_1; t; p)f)(\dots, t^{\pm 1/2}z_i, \dots)\Delta^{(n)}(; t^{\pm 1/2}t_0, t^{\pm 1/2}u_0, v_0, v_1, v_2, v_3; t^2; p, q) \tag{4.47}$$

is invariant under $(v_0, v_1, v_2, v_3) \to (v_2', v_3', v_0', v_1')$. After specializing the image of the difference operator, f appears in principle in 4^n different specializations: corresponding to each z_i is a pair of arguments, one of

$$(q^{-1/2}t^{-1/2}z_i, q^{-1/2}t^{1/2}z_i), (q^{-1/2}t^{-1/2}z_i, q^{1/2}t^{1/2}z_i), (q^{1/2}t^{-1/2}z_i, q^{-1/2}t^{1/2}z_i), (q^{1/2}t^{-1/2}z_i, q^{1/2}t^{1/2}z_i). \tag{4.48}$$

The third pair never actually occurs (the coefficient vanishes), and we can arrange to combine the first and fourth cases by shifting the variable by $q^{1/2}$ or $q^{-1/2}$ as appropriate. (This changes the contour of that portion of the integral, but we can move it back without crossing over any poles.) We thus obtain 2^n different specializations of f, involving the pairs

$$(t^{-1/2}z_i, t^{1/2}z_i) \text{ and } (q^{-1/2}t^{-1/2}z_i, q^{1/2}t^{1/2}z_i); \tag{4.49}$$

the coefficient of a given specialization of f is a sum of 2^m terms where m is the number of times the first pair is used. If we fix a given specialization of f, we can remove common factors of the 2^m terms of its coefficient to obtain the instance $(p, q, t, v_0, v_1, v_2, v_3) \mapsto (p, qt, t^2, pq^{1/2}t/v_0, pq^{1/2}t/v_1, q^{-1/2}v_2, q^{-1/2}v_3)$ of Lemma 2.15. ☐

Of course, the usual Littlewood identity also comes in a dual form, and the same applies at the elliptic level. Since duality breaks the symmetry between p and q, it in particular does not apply at the level of partition pairs. However, we do obtain the following, purely p-elliptic, identity.

Corollary 4.11. *One has the identity*

$$\sum_\mu \left\langle {2\mu \atop \lambda} \right\rangle_{[a,b];q,t;p} \frac{\Delta_\mu(a|v_0, v_1, v_2, v_3; q^2, t; p)}{\Delta_\lambda(a/b|v_0, v_1, v_2, v_3; q, t; p)} \propto \sum_\mu \left\langle {\lambda \atop 2\mu} \right\rangle_{[a/b,b];q,t;p} \frac{\Delta_\mu(a/b^2|v_0, v_1, v_2, v_3; q^2, t; p)}{\Delta_{2\mu}(a/b^2|v_0, v_1, v_2, v_3; q, t; p)}, \tag{4.50}$$

subject to the balancing condition

$$v_0v_1v_2v_3b^2 = pqta^2, \tag{4.51}$$

and the termination conditions

$$t^{\mathbb{N}} \cap \{v_0, v_1, v_2, v_3, b, bq\} \neq \emptyset \tag{4.52}$$

$$q^{-2\mathbb{N}} \cap \{v_0, v_1, v_2, v_3, b, bq\} \neq \emptyset, \tag{4.53}$$

with associated conditions on λ. The constant is given by the value for $\lambda = 0$.

Analytically continuing the left-hand side to an integral produces the following dual vanishing integral.

Corollary 4.12. *If $t^{2n-2}t_0t_1t_2t_3u_0^2 = pq$, then*

$$\langle \tilde{R}^{(n)}_{\boldsymbol{\lambda}}(;t_0{:}t_1,t_2,t_3;u_0,u_0;q,t;p)\rangle_{t_0,t_1,t_2,t_3,u_0,qu_0;t;p,q^2} \tag{4.54}$$

vanishes unless λ is of the form 2μ, when the integral is

$$\frac{\Delta_\mu(1/u_0^2|t^n, t^{n-1}t_0t_1, t^{n-1}t_0t_2, t^{n-1}t_0t_3, 1/t^{n-1}t_0u_0, q/t^{n-1}t_0u_0; q^2, t; p)}{\Delta_{2\mu}(1/u_0^2|t^n, t^{n-1}t_0t_1, t^{n-1}t_0t_2, t^{n-1}t_0t_3, 1/t^{n-1}t_0u_0, 1/t^{n-1}t_0u_0; q, t; p)}. \tag{4.55}$$

Again, here, the vanishing corresponds to the fact that evaluation at a partition with respect to q^2, t, is also evaluation at the doubled partition with respect to q, t:

$$(q^2)^{\mu_i}t^{n-i}t_0 = q^{2\mu_i}t^{n-i}t_0. \tag{4.56}$$

This continues to hold even for partition pairs:

$$p^{\lambda_i}(q^2)^{\mu_i}t^{n-i}t_0 = p^{\lambda_i}q^{2\mu_i}t^{n-i}t_0, \tag{4.57}$$

suggesting the conjecture that

$$\langle \tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\lambda}}(;t_0{:}t_1,t_2,t_3;u_0,u_0;t;p,q)\rangle^{(n)}_{t_0,t_1,t_2,t_3,u_0,qu_0;t;p,q^2} \tag{4.58}$$

vanishes unless $\boldsymbol{\lambda} = (1,2)\boldsymbol{\mu}$ for some partition pair $\boldsymbol{\mu}$ (where $(1,2)(\mu,\nu) = (\mu, 2\nu)$), when it equals

$$\frac{\Delta_{\boldsymbol{\mu}}(1/u_0^2|t^n, t^{n-1}t_0t_1, t^{n-1}t_0t_2, t^{n-1}t_0t_3, 1/t^{n-1}t_0u_0, q/t^{n-1}t_0u_0; t; p, q^2)}{\Delta_{(1,2)\boldsymbol{\mu}}(1/u_0^2|t^n, t^{n-1}t_0t_1, t^{n-1}t_0t_2, t^{n-1}t_0t_3, 1/t^{n-1}t_0u_0, 1/t^{n-1}t_0u_0; t; p, q)}. \tag{4.59}$$

Note, however, that this does not correspond to a vanishing result with respect to the other partition.

The transformation analogue of this extended conjecture is the following.

Conjecture L2 ($q^{1/2}$). *If $t^{2n-2}t_0^2u_0^2v_0v_1v_2v_3 = p^2q^2$, then*

$$\begin{aligned}
&\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(;t_0,u_0;t;p,q)\Delta^{(n)}(;t_0,qt_0,u_0,qu_0,v_0,v_1,v_2,v_3;t;p,q^2)\\
&\qquad = \prod_{0\le r\le 3}\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0/u_0|t^{n-1}t_0v_r;t;p,q)\prod_{1\le i\le n}\Gamma_{p,q}(t^{n-i}t_0v_r, t^{n-i}u_0v_r)\\
&\qquad\quad \times\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(;t_0,u_0;t;p,q)\Delta^{(n)}(;t_0,qt_0,u_0,qu_0,v'_0,v'_1,v'_2,v'_3;t;p,q^2),
\end{aligned} \tag{4.60}$$

where $v'_r = pq/t^{n-1}t_0u_0v_r$.

Remark. If $\boldsymbol{\lambda} = (l,m)^n$ (so in particular if $n=1$), or $t^n t_0 u_0 = pq$, this is again a special case of the transformation of [15, §9]. When $t=q$ or $t=p$, this again essentially reduces to a pfaffian, except that the individual entries include cases with $n=2$, so this does not quite lead to a proof in that case. Note that in this case, the cross-terms do not quite cancel, so each term involves the factor

$$\prod_{1\le i<j\le n} \frac{q^{1/2} z_i^{-1}\theta_{q^2}(z_i z_j^{\pm 1})}{\theta_{q^2}(q z_i z_j^{\pm 1})}. \tag{4.61}$$

Since ([12, Thm. 2.10])

$$\mathrm{pf}_{1\le i<j\le 2n} \frac{q^{1/2} z_i^{-1}\theta_{q^2}(z_i z_j^{\pm 1})}{\theta_{q^2}(q z_i z_j^{\pm 1})} = \prod_{1\le i<j\le 2n} \frac{q^{1/2} z_i^{-1}\theta_{q^2}(z_i z_j^{\pm 1})}{\theta_{q^2}(q z_i z_j^{\pm 1})}, \tag{4.62}$$

and similarly for odd n, one can adapt the argument of [2]. Indeed, one finds that for all $n\ge 0$,

$$\prod_{1\le i<j\le n} \frac{q^{1/2} z_i^{-1}\theta_{q^2}(z_i z_j^{\pm 1})}{\theta_{q^2}(q z_i z_j^{\pm 1})} = \frac{1}{2^{\lfloor n/2\rfloor}\lfloor n/2\rfloor!}\sum_{\pi\in S_n}\sigma(\pi)\pi\cdot \prod_{1\le i\le \lfloor n/2\rfloor} \frac{q^{1/2} z_{2i-1}^{-1}\theta_{q^2}(z_{2i-1} z_{2i}^{\pm 1})}{\theta_{q^2}(q z_{2i-1} z_{2i}^{\pm 1})}, \tag{4.63}$$

where $\pi\in S_n$ acts by permuting the variables. Since the remainder of the integrand is antisymmetric, each term in this sum has the same integral, so that one again obtains a sum of products of low-dimensional integrals.

Proposition 4.13. *Conjecture L2 holds if $t^{n-1}t_0u_0v_0v_1/pq \in \{1, 1/p, 1/q, t\}$.*

Proof. Again, the case $t^{n-1}t_0u_0v_0v_1/pq = 1$ is trivial. The case $t^{n-1}t_0u_0v_0v_1/pq = 1/q$ corresponds to the fact that

$$\int \bigl(D_q^{(n)}(t_0,u_0,v_0;t;q)f\bigr)\Delta^{(n)}(t_0,qt_0,u_0,qu_0,v_0,v_1,v_2,v_3;t;p,q^2) \tag{4.64}$$

is invariant under $(v_0,v_1,v_2,v_3)\to(v_2',v_3',v_0',v_1')$ for any function f. Expanding this as a sum of 2^n terms and undoing all variable shifts gives a manifestly invariant sum; indeed, changing the v parameters has the same effect as inverting all the variables.

Similarly,

$$\int \bigl(D_p^{(n)}(t_0,u_0,v_0;t;q)f\bigr)\Delta^{(n)}(t_0,qt_0,u_0,qu_0,v_0,v_1,v_2,v_3;t;p,q^2) \tag{4.65}$$

is invariant; after expanding and unshifting, one obtains the special case of Lemma 2.15 with

$$(p,q,t,v_0,v_1,v_2,v_3)\mapsto(q^2,p/q,t,p^{1/2}q/v_0,p^{1/2}q/v_1,p^{-1/2}v_2,p^{-1/2}v_3). \tag{4.66}$$

When $t^{n-1}t_0u_0v_0v_1/pq = t$, we can use the integral equation to write

$$\begin{aligned}\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(;t_0,u_0;t;p,q) = {}&\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0/u_0|t^{n-1}t_0v_0,t^{n-1}t_0v_1;t;p,q)\\ &\times \mathcal{I}^{(n)}(t^{-1/2}u_0{:}t^{-1/2}t_0,t^{-1/2}v_0;p,q)\,\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(;t^{-1/2}t_0,t^{-1/2}u_0;t;p,q)\end{aligned} \tag{4.67}$$

on the right-hand side. After changing order of integration, the inner integral has the form

$$I\!I^{(n-1)}_n(pq/v_0,pq/v_1,v_2,v_3,\dots,t^{1/2}x_i^{\pm 1},qt^{1/2}x_i^{\pm 1},\dots;pq/t;p,q^2), \tag{4.68}$$

so can be transformed using Theorem 2.14 to give the desired identity. ∎

Remark. It is natural to try to extend the proofs for $1/p$ and $1/q$ using the iterated difference operators introduced in the proof of [15, Thm. 9.7]. We find that the proof in those cases would reduce to showing that when $v_0v_1v_2v_3 = p^2q^2$,

$$\prod_{\substack{1\le i\le n\\0\le r<4}} \frac{1}{\Gamma_{p,q^2}(v_r x_i^{\pm 1})} D^{(n)}_{l,m}(t;p,q) \prod_{\substack{1\le i\le n\\0\le r<4}} \Gamma_{p,q^2}(p^{l/2}q^{m/2}v_r x_i^{\pm 1}) \frac{\Delta^{(n)}(x_1,\dots,x_n;t;p,q^2)}{\Delta^{(n)}(x_1,\dots,x_n;t;p,q)} \tag{4.69}$$

is invariant under $(v_0,v_1,v_2,v_3)\mapsto(pq/v_0,pq/v_1,pq/v_2,pq/v_3)$. The case $(l,m)=(0,1)$ holds even without balancing condition (or limit on the number of v_r factors). Using this, one can mimic the proof of Theorem 4.5 above to show that the identity for (l,m) implies the identity for $(l,m+1)$; this implies Conjecture L2 whenever $t^{n-1}t_0u_0v_0v_1/pq \in \{q^{-m},p^{-1}q^{-m}\}$ for $m\ge 0$. Note that when $m=0$, the ratio of any two terms of the sum is q^2-elliptic, so this reduces to an algebraic statement, presumably equivalent to the conjectured equation (4.71) below. It should be possible to interpret and extend the proof of Proposition 4.10 in a similar way, with the corresponding identity an analytic continuation of Corollary 4.6, though the specialization of the variables makes this nontrivial.

Corollary 4.14. *If $t^{2n-2}t_0^2u_0^2v_0v_1 = p$, then*

$$\begin{aligned}&\left\langle \frac{\mathcal{R}^{*(n)}_{\boldsymbol\lambda}(;t_0,u_0;t;p,q)}{\Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0/u_0|t^{n-1}t_0v_0,t^{n-1}t_0v_1;t;p,q)}\right\rangle^{(n)}_{t_0,qt_0,u_0,qu_0,v_0,v_1;t;p,q^2}\\ &= \Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0/u_0|qt^{n-2}t_0^2;t;p,q)\sum_{\boldsymbol\mu}\left\langle \begin{matrix}\boldsymbol\lambda\\(1,2)\boldsymbol\mu\end{matrix}\right\rangle_{[t^{n-1}t_0/u_0,t^{n-1}t_0u_0];t;p,q}\frac{\Delta_{\boldsymbol\mu}(1/u_0^2|t^n,qt^{n-1}t_0^2;t;p,q^2)}{\Delta_{(1,2)\boldsymbol\mu}(1/u_0^2|t^n,qt^{n-1}t_0^2;t;p,q)}.\end{aligned} \tag{4.70}$$

Proof. The left-hand side is simply the instance $v_2v_3 = pq$ of the left-hand side of Conjecture L2, which is manifestly invariant of the choice of v_2. In particular, we may arrange for $t^{n-1}t_0u_0v_0v_2\in\{p,q\}$ so that we may apply our known special cases of the conjecture. In these cases, the transformation simply applies a p- or q-shift to v_0 and v_1; since the left-hand side is meromorphic in v_0/v_1 and invariant under both p and q-shifts, it is in fact independent of v_0/v_1. Taking the limit $v_0\to 1/t^{n-1}u_0$ gives the desired result. ∎

The "q-elliptic" half[1] of the dual Littlewood identity reads (after swapping p and q)

$$\sum_\mu \left\langle\begin{matrix}\mu\\\lambda\end{matrix}\right\rangle_{[a,b];q,t;p}\frac{\Delta_\mu(a|v_0,v_1,v_2,v_3;q,t;p^2)}{\Delta_\lambda(a/b|v_0,v_1,v_2,v_3;q,t;p)} \propto^? \sum_\mu \left\langle\begin{matrix}\lambda\\\mu\end{matrix}\right\rangle_{[a/b,b];q,t;p}\frac{\Delta_\mu(a/b^2|v_0,v_1,v_2,v_3;q,t;p^2)}{\Delta_\mu(a/b^2|v_0,v_1,v_2,v_3;q,t;p)}, \tag{4.71}$$

and in particular involves both p-abelian and p^2-abelian functions. This transformation is taken to itself by duality, but if we use a modular transformation, we can replace the 2-isogeny

$$z\in\mathbb{C}^*/\langle p^2\rangle \mapsto z\in\mathbb{C}^*/\langle p\rangle \tag{4.72}$$

by

$$z\in\mathbb{C}^*/\langle p^{1/2}\rangle \mapsto z^2\in\mathbb{C}^*/\langle p\rangle. \tag{4.73}$$

This then gives rise to the following conjecture, upon lifting back to an integral transformation.

[1] To be precise, this is only q^2-elliptic, but we abuse terminology to distinguish it from the p-elliptic half.

Conjecture L3 (-1)**.** *If* $t^{n-1}t_0^2u_0^2v_0v_1v_2v_3 = pq$, *then*

$$\begin{aligned}
&\int \mathcal{R}^{*(n)}_{\boldsymbol\lambda}(\dots,z_i^2,\dots;t_0^2,u_0^2;t;p,q)\Delta^{(n)}(;t_0,-t_0,u_0,-u_0,v_0,v_1,v_2,v_3;t^{1/2};p^{1/2},q^{1/2})\\
&\quad= \prod_{0\le r\le 3}\Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0^2/u_0^2|t^{n-1}t_0^2v_r^2;t;p,q)\prod_{0\le i<n}\Gamma_{p,q}(t^it_0^2v_r^2,t^iu_0^2v_r^2)\\
&\qquad\times\int \mathcal{R}^{*(n)}_{\boldsymbol\lambda}(\dots,z_i^2,\dots;t_0^2,u_0^2;t;p,q)\Delta^{(n)}(;t_0,-t_0,u_0,-u_0,v_0',v_1',v_2',v_3';t^{1/2};p^{1/2},q^{1/2}),
\end{aligned}\tag{4.74}$$

where $v_r' = p^{1/2}q^{1/2}/t^{(n-1)/2}t_0u_0v_r$.

Again, this holds if $\boldsymbol\lambda = (l,m)^n$ or $t^{n/2}t_0u_0 = p^{1/2}q^{1/2}$. When $t=p$ or $t=q$ (really four cases, as either square root of t will work), the integral has a similar structure to the pfaffian case of Conjecture L2, except that the pfaffian factor is now either

$$\prod_{1\le i<j\le n}\frac{\theta_{p^{1/2}}(z_iz_j^{\pm 1})}{\theta_{p^{1/2}}(-z_iz_j^{\pm 1})}\tag{4.75}$$

or

$$\prod_{1\le i<j\le n}\frac{\theta_{q^{1/2}}(z_iz_j^{\pm 1})}{\theta_{q^{1/2}}(-z_iz_j^{\pm 1})}\tag{4.76}$$

(This is a pfaffian by virtue of being a modular transformation of the previous pfaffian.)

Although it does not have an associated vanishing result per se, there is an associated analogue of Corollary 4.9, namely the conjecture that if $t^{n-1}t_0t_1t_2t_3u_0^2 = -p^{1/2}q^{1/2}$, then

$$\begin{aligned}
&\langle \tilde{\mathcal{R}}^{(n)}_{\boldsymbol\lambda}(\dots,z_i^2,\dots;t_0^2{:}t_1^2,t_2^2,t_3^2;u_0^2,u_0^2;t;p,q)\rangle^{(n)}_{t_0,t_1,t_2,t_3,u_0,-u_0;t^{1/2};p^{1/2},q^{1/2}}\\
&\quad=\frac{\Delta_{\boldsymbol\lambda}(1/u_0^2|t^{n/2},t^{(n-1)/2}t_0t_1,t^{(n-1)/2}t_0t_2,t^{(n-1)/2}t_0t_3,1/t^{(n-1)/2}t_0u_0,-1/t^{(n-1)/2}t_0u_0;t^{1/2};p^{1/2},q^{1/2})}{\Delta_{\boldsymbol\lambda}(1/u_0^4|t^n,t^{n-1}t_0^2t_1^2,t^{n-1}t_0^2t_2^2,t^{n-1}t_0^2t_3^2,1/t^{n-1}t_0^2u_0^2,1/t^{n-1}t_0^2u_0^2;t;p,q)}.
\end{aligned}\tag{4.77}$$

If we eliminate u_0 using the balancing condition then take the limit $p\to 0$, we obtain a conjectural integral of Koornwinder polynomials which in the notation of [13] reads

$$\begin{aligned}
&I_K(\tilde K_\lambda([p_{2k}];q^2,t^2,T^2;t_0^2,t_1^2,t_2^2,t_3^2);q,t,T;t_0,t_1,t_2,t_3)\\
&\quad=\frac{(-1)^{|\lambda|}C^0_\lambda(T,Tt_0t_1t_2t_3/t^2;q,t)C^+_\lambda(T^2t_0t_1t_2t_3/t^3;q,t)C^-_\lambda(-q;q,t)\prod_{0\le r<s<4}C^0_\lambda(Tt_rt_s/t;q,t)}{C^0_{2\lambda^2}(T^2t_0t_1t_2t_3/t^2;q,t)C^+_\lambda(-T^2t_0t_1t_2t_3/qt^2;q,t)C^-_\lambda(t;q,t)}
\end{aligned}\tag{4.78}$$

(When $T=t^n$, this is an n-dimensional integral of n-variable Koornwinder polynomials evaluated at $z_1^2,\dots,z_n^2$.) This is no longer related to a Littlewood identity, but is instead related to an identity conjectured by Kawanaka in [8] and proved (via elliptic means) in [9]. Indeed, if we set $T=0$ (i.e., take $n\to\infty$) we obtain an identity which can be used to integrate the left-hand side of the appropriate Cauchy identity [13, Thm. 7.19] term by term. Since the coefficients in that Cauchy identity are Macdonald polynomials, we obtain the sum

$$\sum_\lambda \frac{C^-_\lambda(-t^{1/2};q^{1/2},t^{1/2})}{C^-_\lambda(q^{1/2};q^{1/2},t^{1/2})}P_\lambda(\dots,x_i,\dots;q,t) = \prod_j\frac{(-t^{1/2}x_j;q^{1/2})}{(x_j;q^{1/2})}\prod_{j<k}\frac{(tx_jx_k;q)}{(x_jx_k;q)},\tag{4.79}$$

using [13, Thm. 7.17] to compute the integral and obtain the right-hand side. Note that since this is independent of t_0, t_1, t_2, t_3, and the case $t_1=-t_0$ of the elliptic conjecture follows from Corollary 4.16 below, this argument gives a second proof of Kawanaka's conjecture.

Proposition 4.15. *Conjecture L3 holds if* $t^{n-1}t_0^2u_0^2v_0^2v_1^2/pq \in \{1, 1/p, 1/q, t\}$.

Proof. Again, the 1 case is trivial, and by symmetry the $1/p$ and $1/q$ cases are equivalent. The $1/q$ case reduces to the invariance of

$$\int (D_q^{(n)}(t_0^2, u_0^2, v_0^2; t; p)\, \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(; q^{1/2}t_0^2, q^{1/2}u_0^2; t; p, q))(\dots, z_i^2, \dots)$$
$$\times \Delta^{(n)}(; t_0, -t_0, u_0, -u_0, v_0, v_1, v_2, v_3; t^{1/2}; p^{1/2}, q^{1/2}) \tag{4.80}$$

under $v_0, v_1, v_2, v_3 \to v_2', v_3', v_0', v_1'$. This follows as above, this time via the special case $(p, q, t, v_0, v_1, v_2, v_3) \mapsto (p^{1/2}, -q^{1/2}, t^{1/2}, -p^{1/2}q^{1/4}/v_0, -p^{1/2}q^{1/4}v_1, q^{-1/4}v_2, q^{-1/4}v_3)$ of Lemma 2.15.

Similarly, the case $t^{n-1}t_0^2u_0^2v_0^2v_1^2/pq = t$ reduces to a special case of Theorem 2.14 as before. ☐

Remark. As in the remark following Proposition 4.13, the identity used in the proof for $1/p$ and $1/q$ can be interpreted as a special case of a more general difference equation. To wit, if $v_0v_1v_2v_3 = pq$, then

$$\prod_{\substack{1\le i\le n\\ 0\le r<4}} \frac{1}{\Gamma_{p^{1/2},q^{!/2}}(v_r x_i^{\pm 1/2})} D^{(n)}_{l,m}(t; p, q) \prod_{\substack{1\le i\le n\\ 0\le r<4}} \Gamma_{p^{1/2},q^{!/2}}(S_{l,m}^{1/4} v_r x_i^{\pm 1/2}) \frac{\Delta^{(n)}(x_1^{1/2}, \dots, x_n^{1/2}; t^{1/2}; p^{1/2}, q^{1/2})}{\Delta^{(n)}(x_1, \dots, x_n; t; p, q)} \tag{4.81}$$

should be invariant under $(v_0, v_1, v_2, v_3) \mapsto (-p^{1/2}q^{1/2}/v_0, -p^{1/2}q^{1/2}/v_1, -p^{1/2}q^{1/2}/v_2, -p^{1/2}q^{1/2}/v_3)$. Here, one must be careful to make consistent choices of fourth roots of p and q so that the action of the difference operator is still well-defined despite having taken square roots of the variables.

Corollary 4.16. *If* $t^{n-1}t_0^2u_0^2v_0v_1 = p^{1/2}q^{1/2}$, *then*

$$\left\langle \frac{\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots, z_i^2, \dots; t_0^2, u_0^2; t; p, q)}{\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0^2/u_0^2 | t^{n-1}t_0^2v_0^2, t^{n-1}t_0^2v_1^2; t; p, q)} \right\rangle^{(n)}_{t_0, -t_0, u_0, -u_0, v_0, v_1; t^{1/2}; p^{1/2}, q^{1/2}}$$
$$= \Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0^2/u_0^2 | t^{n-1}t_0^4; t; p, q) \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{matrix} \right\rangle_{[t^{n-1}t_0^2/u_0^2, t^{n-1}t_0^2u_0^2]; t; p, q} \frac{\Delta_{\boldsymbol{\mu}}(1/u_0^2 | t^{n/2}, -t^{(n-1)/2}t_0^2; t^{1/2}; p^{1/2}, q^{1/2})}{\Delta_{\boldsymbol{\mu}}(1/u_0^4 | t^n, t^{n-1}t_0^4; t; p, q)}. \tag{4.82}$$

5 Quadratic transformations

In [17], there were several other integrals that vanished unless a given partition (or its conjugate) was of the form μ^2. If we restate the integrals in terms of interpolation polynomials rather than Koornwinder polynomials, the right-hand side becomes a sum over binomial coefficients $\binom{\lambda}{\mu^2}$, multiplied by the nonzero values, suggesting that it should be a special case of the right-hand side of Theorem 4.7. For most of the results of [17], the nonzero values were not established, but in the case of Theorem 4.8 op. cit., they are known, and one can thus use Theorem 4.7 as a guide to formulating the following conjecture.

Conjecture 1. *For generic parameters satisfying* $t^{4n-1}t_0^2t_1^2u_0^2 = pq$, *the integral*

$$\langle \tilde{\mathcal{R}}^{(2n)}_{\boldsymbol{\lambda}}(\dots, \pm\sqrt{-z_i}, \dots; t_0\sqrt{-1}{:} - t_0\sqrt{-1}, t_1\sqrt{-1}, -t_1\sqrt{-1}; u_0\sqrt{-1}, -tu_0\sqrt{-1}; t; p, q)\rangle^{(n)}_{t_0^2, t_1^2, u_0^2, t, pt, qt; t^2; p^2, q^2} \tag{5.1}$$

vanishes unless $\boldsymbol{\lambda}$ is of the form $\boldsymbol{\mu}^2$, in which case the integral is

$$\frac{\Delta_{\boldsymbol{\mu}}(1/t^2u_0^2|t^{2n},t^{2n-1}t_0^2,1/t^{2n-1}t_0u_0,1/t^{2n}t_0u_0;t^2;p,q)}{\Delta_{\boldsymbol{\mu}^2}(1/tu_0^2|t^{2n},t^{2n-1}t_0^2,1/t^{2n-1}t_0u_0,1/t^{2n}t_0u_0;t;p,q)}. \tag{5.2}$$

If one solves for u_0 in the balancing condition then lets $p \to 0$, this naturally becomes Theorem 4.8 of [17] (and agrees with the nonzero values computed in [13]); one also notes that the conjecture is consistent under negating t_0 or swapping t_0 and t_1.

There is an alternate formulation of this conjecture as an identity of "hypergeometric" sums. The key observation is that the Cauchy-type interpolation function satisfies the transformation

$$\mathcal{R}^{*(2n)}_{\boldsymbol{\lambda}}(\dots,\pm\sqrt{-z_i},\dots;pq/t^{2n}\sqrt{-1}u_0,\sqrt{-1}u_0;t;p,q) = \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,z_i,\dots;p^2q^2/t^{2n}u_0^2,u_0^2;t^2;p^2,q^2), \tag{5.3}$$

which follows immediately from the product formula for such functions. (Recall also that, as remarked after Theorem 2.5, we may freely extend the right-hand side to the case $\ell(\boldsymbol{\lambda}) > n$, without invalidating our further computations.)

Thus if the conjecture holds, one can compute the integral

$$\langle \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(;p^2q^2/t^{2n}u_0^2,u_0^2;t^2;p^2,q^2)\rangle^{(n)}_{t_0^2,t_1^2,u_0^2,t,pt,qt;t^2,p^2,q^2} \tag{5.4}$$

in two different ways: either by expanding

$$\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(;p^2q^2/t^{2n}u_0^2,u_0^2;t^2;p^2,q^2) = \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[p^2q^2/t^2u_0^4,p^2q^2/t^{2n}t_0^2u_0^2](p^2q^2t_0^2/t^2u_0^2);t^2;p^2,q^2} \mathcal{R}^{*(n)}_{\boldsymbol{\mu}}(;t_0^2,u_0^2;t^2;p^2,q^2) \tag{5.5}$$

and applying the elliptic analogue of Kadell's lemma [15, Cor. 9.3], or by expanding

$$\begin{aligned}&\mathcal{R}^{*(2n)}_{\boldsymbol{\lambda}}(;pq/t^{2n}\sqrt{-1}u_0,\sqrt{-1}u_0;t;p,q)\\ &\qquad = \sum_{\boldsymbol{\mu}} \left\langle \begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[-pq/tu_0^2,-pq/t^{2n}t_0u_0](pqt_0/tu_0);t;p,q} \mathcal{R}^{*(2n)}_{\boldsymbol{\mu}}(;t_0\sqrt{-1},u_0\sqrt{-1};t;p,q)\end{aligned} \tag{5.6}$$

and applying the conjecture in the form

$$\begin{aligned}&\langle R^{*(2n)}_{\boldsymbol{\mu}}(\dots,\pm\sqrt{-z_i},\dots;t_0\sqrt{-1},u_0\sqrt{-1};t;p,q)\rangle^{(2n)}_{t_0^2,t_1^2,u_0^2,t,pt,qt;t^2,p^2,q^2}\\ &= \Delta^0_{\boldsymbol{\mu}}(t^{2n-1}t_0/u_0|t^{2n-1}t_0^2,\pm t^{2n-1}t_0t_1;t;p,q)\sum_{\boldsymbol{\nu}} \left\langle \begin{matrix}\boldsymbol{\mu}\\ \boldsymbol{\nu}^2\end{matrix}\right\rangle_{[t^{2n-1}t_0/u_0,t^{2n}t_0u_0];t;p,q} \frac{\Delta_{\boldsymbol{\nu}}(1/t^2u_0^2|t^{2n},t^{2n-1}t_0^2;t^2;p,q)}{\Delta_{\boldsymbol{\nu}^2}(1/tu_0^2|t^{2n},t^{2n-1}t_0^2;t;p,q)}.\end{aligned} \tag{5.7}$$

One thus finds that the conjecture implies

$$\begin{aligned}&\sum_{\boldsymbol{\mu}} \left\langle \begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[p^2q^2/t^2u_0^4,p^2q^2/t^{2n}t_0^2u_0^2](p^2q^2t_0^2/t^2u_0^2);t^2;p^2,q^2} \Delta^0_{\boldsymbol{\mu}}(t^{2n-2}t_0^2/u_0^2|t^{2n-2}t_0^2t_1^2,t^{2n-1}t_0^2,t^{2n-1}pt_0^2,t^{2n-1}qt_0^2;t^2,p^2,q^2)\\ &\quad = \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \Delta^0_{\boldsymbol{\mu}}(t^{2n-1}t_0/u_0|t^{2n-1}t_0^2,\pm t^{2n-1}t_0t_1;t;p,q)\frac{\Delta_{\boldsymbol{\nu}}(1/t^2u_0^2|t^{2n},t^{2n-1}t_0^2;t^2;p,q)}{\Delta_{\boldsymbol{\nu}^2}(1/tu_0^2|t^{2n},t^{2n-1}t_0^2;t;p,q)}\\ &\qquad\quad \times \left\langle \begin{matrix}\boldsymbol{\lambda}\\ \boldsymbol{\mu}\end{matrix}\right\rangle_{[-pq/tu_0^2,-pq/t^{2n}t_0u_0](pqt_0/tu_0);t;p,q} \left\langle \begin{matrix}\boldsymbol{\mu}\\ \boldsymbol{\nu}^2\end{matrix}\right\rangle_{[t^{2n-1}t_0/u_0,t^{2n}t_0u_0];t;p,q}.\end{aligned} \tag{5.8}$$

In fact, this is equivalent to the conjecture, since one can equally well expand the biorthogonal functions in Cauchy-type interpolation functions. Note also that n enters here only via t^{2n}, and thus one can analytically continue in this extra parameter.

Proposition 5.1. *Conjecture 1 holds whenever $\ell(\boldsymbol{\lambda}) \le 1$.*

Proof. By triangularity, it suffices to prove (5.8) when $\ell(\boldsymbol{\lambda}) \le 1$. On the right-hand side, this forces $\ell(\boldsymbol{\mu}) \le 1$, so $\ell(\boldsymbol{\nu}^2) \le 1$, and thus $\boldsymbol{\nu} = 0$, making the double sum on the right collapse to a single sum. The resulting identity of univariate elliptic hypergeometric sums is a special case of a known quadratic transformation [22, Thm. 5.1] (the discrete version of Proposition 5.12 below). $\square$

The integral analogue of Conjecture 1 appears to be the following. The label "$(-1, t^{-1/2})$" and similar labels below will be explained at the end of the section. For consistency with the later conjectures, we replace $2n$ by n but insist that n be even.

Conjecture Q1 $(-1, t^{-1/2})$**.** *For otherwise generic parameters satisfying $t^n t_0 t_1 t_2 u_0 = -pq$, and even $n \ge 0$, one has*

$$\begin{aligned}
&\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots, \pm\sqrt{-z_i}, \dots; t_0\sqrt{-1}, u_0\sqrt{-1}; t; p, q)\Delta^{(n/2)}(\dots, z_i, \dots; t_0^2, t_1^2, t_2^2, u_0^2, t, pt, qt, pqt; t^2; p^2, q^2)\\
&\quad= \frac{\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0/u_0| - t^{n-1}t_0t_1; t; p, q)}{\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0/u_0|t^n t_0 t_1; t; p, q)} \prod_{0\le i<n} \prod_{0\le r<s<3} \frac{\Gamma_{p,q}(-t^i t_r t_s)}{\Gamma_{p,q}(t^{i+1} t_r t_s)}\\
&\qquad\times \int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots, t^{\pm 1/2} z_i, \dots; t^{1/2} t_0, t^{1/2} u_0; t; p, q)\Delta^{(n/2)}(\dots, z_i, \dots; t_0, tt_0, t_1, tt_1, t_2, tt_2, u_0, tu_0; t^2; p, q).
\end{aligned} \tag{5.9}$$

Remark. Note that the Δ^0 factor is symmetric under swapping t_1 and t_2, since the balancing condition makes $(-t^{n-1}t_0t_1)(t^n t_0 t_2) = pqt^{n-1}t_0/u_0$.

If we use the connection coefficient formula [14, Cor. 4.14] to expand the interpolation functions on each side in terms of the corresponding functions with t_0 replaced by t_1, the result is a linear combination of instances of the conjecture with t_0 and t_1 swapped. The conjecture is similarly consistent under $\boldsymbol{\lambda} \mapsto (l, m)^n + \boldsymbol{\lambda}$, and (combining the two) under shifts in t_1, t_2, expanding via the Pieri identity, [14, Thm. 4.17], or equivalently the special case $\boldsymbol{\lambda} = 0$, $m = 1$, $w_0 w_1 \in p^{-\mathbb{N}} q^{-\mathbb{N}}$ of the skew Cauchy identity, Theorem 3.7 above. In particular, the case $t_2 = p^l q^m \sqrt{pq/t}$ reduces to the case $t_2 = \sqrt{pq/t}$, which in turn via Theorem 4.7 reduces to Conjecture 1. This, in fact, was how the above conjecture was formulated, by analytically continuing the result of shifting parameters and applying the Pieri identity. The fact that the resulting transformation is symmetric in t_1, t_2 is a reassuring consistency; the fact that the right-hand side appears not to be symmetrical under $t_1, t_2 \mapsto -t_1, -t_2$ is less reassuring, but in fact a special case of Conjecture L1 would restore this symmetry.

This conjecture satisfies a further consistency condition. Consider the linear functional on the space spanned by the (linearly independent) functions $\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(; t_0\sqrt{-1}, u_0\sqrt{-1}; t; p, q)$ defined by taking

$$\sum_{\boldsymbol{\lambda}} c_{\boldsymbol{\lambda}} \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(; t_0\sqrt{-1}, u_0\sqrt{-1}; t; p, q) \tag{5.10}$$

to the sum

$$\sum_{\boldsymbol\lambda} c_{\boldsymbol\lambda} \frac{\Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0/u_0|-t^{n-1}t_0t_1;t;p,q)}{\Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0/u_0|t^n t_0t_1;t;p,q)} \mathcal{R}^{*(n)}_{\boldsymbol\lambda}(\dots,t^{\pm 1/2}z_i,\dots;t^{1/2}t_0,t^{1/2}u_0;t;p,q) \tag{5.11}$$

and then integrating against the density

$$\Delta^{(n/2)}(\dots,z_i,\dots;t_0,tt_0,t_1,tt_1,t_2,tt_2,u_0,tu_0;t^2;p,q). \tag{5.12}$$

If the conjecture holds, then this linear functional factors through the homomorphism

$$f \mapsto f(\dots,\sqrt{-z_i},\dots) \tag{5.13}$$

and must therefore vanish on the kernel of that homomorphism. It suffices to verify that the image of the functional on the product (and its analogue with p and q swapped)

$$\prod_{1\le i\le n} \frac{\theta_p(v\sqrt{-1}x_i^{\pm 1})}{\theta_p((pq/u_0\sqrt{-1})x_i^{\pm 1})} \mathcal{R}^{*(n)}_{\boldsymbol\lambda}(;t_0\sqrt{-1},u_0\sqrt{-1}/q) \tag{5.14}$$

is invariant under $v\mapsto -v$. The relevant expansion coefficients come from the Pieri identity, and we can recognize the resulting integrand as proportional to

$$D^{+(n)}_q(t^{1/2}u_0{:}t^{1/2}t_0{:}t^{1/2}t_1,t^{1/2}t_2,-t^{-1/2}v;t;p)\, \mathcal{R}^{*(n)}_{\boldsymbol\lambda}(;q^{1/2}t^{1/2}t_0,q^{-1/2}t^{1/2}u_0;t;p,q). \tag{5.15}$$

Since $\boldsymbol\lambda$ was arbitrary, we conclude that for consistency, we need

$$\begin{aligned}\int (\mathcal{D}^{+(n)}_q(t^{1/2}u_0{:}t^{1/2}t_0,t^{1/2}t_1,t^{1/2}t_2,-t^{-1/2}v;t;p)f)(\dots,t^{\pm 1/2}z_i,\dots)&\\ \Delta^{(n/2)}(\dots,z_i,\dots;t_0,tt_0,t_1,tt_1,t_2,tt_2,u_0,tu_0;t^2;p,q)&\end{aligned} \tag{5.16}$$

to be invariant under $v\mapsto -v$ for any function f in the span of the interpolation functions. But this follows by essentially the same argument as the $1/q$ case of Proposition 4.10. (In fact, this adjointness relation is formally equivalent to a special case of the adjointness relation proved there.)

Proposition 5.2. *Conjecture Q1 holds when $n=2$.*

Proof. Note that this is a nontrivial claim even when $\boldsymbol\lambda=0$, as the two integrands involve different values of p and q. However, we observe that in general the case of the conjecture with $\boldsymbol\lambda=(l,m)^n+\boldsymbol\mu$ reduces to the case with $\boldsymbol\lambda=\boldsymbol\mu$, so for $n=2$, it suffices to consider the case $\ell(\boldsymbol\lambda)\le 1$. But then the integral representation of [15] implies the following expression.

$$\begin{aligned}&\mathcal{R}^{*(2)}_{(l,m)}(\pm\sqrt{-z};v\sqrt{-1},u_0\sqrt{-1};t;p,q)\\ &\quad= \frac{\Gamma_{p^2,q^2}(t^2u_0^2z^{\pm 1})}{\Gamma_{p^2,q^2}(u_0^2z^{\pm 1},tz^{\pm 1},ptz^{\pm 1},qtz^{\pm 1},pqtz^{\pm 1})} \times \frac{(p;p)(q;q)\Gamma_{p,q}(t^2)}{2\Gamma_{p^2,q^2}(t^2)^2}\\ &\qquad\times \int_{C'} \mathcal{R}^{*(1)}_{(l,m)}(y;t^{1/2}v\sqrt{-1},t^{-1/2}u_0\sqrt{-1};t;p,q) \frac{\Gamma_{p,q}(t^{-1/2}u_0y^{\pm 1}\sqrt{-1})\Gamma_{p^2,q^2}(-tz^{\pm 1}y^{\pm 2})}{\Gamma_{p,q}(t^{3/2}u_0y^{\pm 1}\sqrt{-1})\Gamma_{p,q}(y^{\pm 2})} \frac{dy}{2\pi\sqrt{-1}y}.\end{aligned} \tag{5.17}$$

If we substitute this in and exchange order of integration, the integral over z becomes an instance of the order 0 elliptic beta integral (the z-dependent factor above cancels five parameters then adds three, making a final total of six), so can be explicitly evaluated, and we thus conclude

$$\begin{aligned}
&\int \mathcal{R}^{*(2)}_{(l,m)}(\pm\sqrt{-z}; v\sqrt{-1}, u_0\sqrt{-1}; t; p, q)\Delta^{(1)}(z; t_0^2, t_1^2, t_2^2, u_0^2, t, pt, qt, pqt; t^2, p^2, q^2) \\
&\quad= \prod_{0\le r<s<3} \Gamma_{p^2,q^2}(t_r^2 t_s^2) \prod_{0\le r<3} \Gamma_{p^2,q^2}(t^2 u_0^2 t_r^2) \\
&\qquad\times \int \mathcal{R}^{*(1)}_{(l,m)}(y; t^{1/2} v\sqrt{-1}, t^{-1/2} u_0\sqrt{-1}; t^2; p, q) \\
&\qquad\quad\times \Delta^{(1)}(y; \pm t^{1/2} t_0\sqrt{-1}, \pm t^{1/2} t_1\sqrt{-1}, \pm t^{1/2} t_2\sqrt{-1}, t^{-1/2} u_0\sqrt{-1}, -t^{3/2} u_0\sqrt{-1}; t^2; p, q),
\end{aligned} \tag{5.18}$$

where we have used the fact that a univariate interpolation function is independent of t. Now, this argument is not actually rigorous, as there are in general difficulties in choosing the contours in allowing the change of variables (except if $l = 0$ or $m = 0$, when there is an open set of parameters allowing both contours to be the unit circle). If $v = -pq/t^2 u_0$, then the interpolation functions can both be written as products, and the result is a special case of Proposition 5.12 below (which can be viewed as the analytic continuation in $p^l q^m$). The corresponding result for $v = t_0$ then follows from the fact that the connection coefficients are the same on both sides.

On the right-hand side, we have

$$\mathcal{R}^{*(2)}_{(l,m)}(t^{\pm 1/2} z; t^{1/2} t_0, t^{1/2} u_0; t; p, q) = \mathcal{R}^{*(1)}_{(l,m)}(z; tt_0, u_0; t^2; p, q), \tag{5.19}$$

and thus the integral on the right-hand side is

$$\int \mathcal{R}^{*(1)}_{(l,m)}(z; tt_0, u_0; t^2; p, q)\Delta^{(1)}(z; t_0, tt_0, t_1, tt_1, t_2, tt_2, u_0, tu_0; t^2; p, q) \tag{5.20}$$

The desired special case of Conjecture Q1 then follows as a special case of [15, Cor. 9.11]. ☐

This immediately implies that Conjecture Q1 holds when $t = 1$. Another special case arises when $t = q$ (or, symmetrically, $t = p$), much as in the discussion following Conjecture L1. Since we no longer have the same interpolation function on both sides of the identity, we need to control the constants somewhat better. Note first that for general t, if we replace the interpolation functions by appropriate versions of

$$F^{(n)}_{\boldsymbol{\lambda}}(z_1, \dots, z_n; t_1, t_0, u_0; t; p, q) := \frac{\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(z_1, \dots, z_n; t_0, u_0; t; p, q)}{\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(t^{n-1} t_1, \dots, t_1; t_0, u_0; t; p, q) \prod_{1\le i\le n} \Gamma_{p,q}(t^{n-i} t_0 t_1, t^{n-i} t_0 u_0, t^{n-i} t_1 u_0)}, \tag{5.21}$$

then this absorbs the constants outside the integrals. With this in mind, we note that

$$F^{(n)}_{\lambda,\mu}(z_1, \dots, z_n; ct_1, ct_0, cu_0; q; p, q) \propto \frac{\sum_{\pi,\rho\in S_n} \sigma(\rho) \prod_{1\le i\le n} F^{(1)}_{\lambda_{\pi_i}, \mu_{\rho_i}+n-\rho_i}(z_i; ct_1, ct_0, cq^{n-1} u_0; q; p, q)}{\prod_{1\le i\le n} \theta_p(cu_0 z_i^{\pm 1}; q)^{-1}_{n-1} \prod_{1\le i<j\le n} c z_i^{-1} \theta_p(z_i z_j^{\pm 1})}, \tag{5.22}$$

where the constant of proportionality is independent of c. (The constant can be explicitly evaluated using Warnaar's determinant identity [26, Lem. 5.3].) As in Conjecture L1, we find that substituting in this expression reduces the $t = q$ case of Conjecture Q1 to a sum of products of instances with $t = q$, $n = 2$ (essentially a pfaffian).

We also have an additional special case when $\boldsymbol{\lambda} = 0$ and in low dimensions (see also [4]).

Proposition 5.3. *If $t \in \{p^{1/2}, q^{1/2}\}$, then Conjecture Q1 holds if either $\boldsymbol{\lambda} = 0$ or $n \leq 6$.*

Proof. First suppose $\boldsymbol{\lambda} = 0$. When $t = q^{1/2}$, two parameters cancel in the left-hand side, allowing it to be evaluated, while the right-hand side can be evaluated by observing that its integrand is equal to an elliptic Selberg integrand of order 0 with $q \mapsto q^{1/2}$; the result follows upon simplifying the gamma factors.

Since the conjecture is consistent under parameter shifts and with respect to the homomorphism $f \mapsto f(\cdots \pm \sqrt{-z_i} \dots)$, we find that we can integrate any function of the form

$$\prod_{1\leq i\leq n/2} \frac{\theta(t_0^2 z_i^{\pm 1}; p^2, q^2)_{l_0,m_0} \theta(t_1^2 z_i^{\pm 1}; p^2, q^2)_{l_1,m_1} \theta(t_2^2 z_i^{\pm 1}; p^2, q^2)_{l_2,m_2}}{\theta((p^2 q^2/u_0^2) z_i^{\pm 1}; p^2, q^2)_{l_0+l_1+l_2, m_0+m_1+m_2}} \tag{5.23}$$

against the left-hand side density in two ways: either directly by shifting parameters in the left-hand side density and applying the $\boldsymbol{\lambda} = 0$ transformation, or indirectly by expanding in images of interpolation functions and transforming term by term. Our consistency conditions show that both approaches will give the same integral (independent of the choice of expansion). Since for $n \leq 6$ the above functions generically span the full image of the space of interpolation functions, we conclude that the transformation actually holds termwise; i.e., Conjecture Q1 holds for all $\boldsymbol{\lambda}$ in this special case. ☐

The above evaluation of the right-hand side generalizes to a transformation, again by observing that both sides have the same integrand.

Proposition 5.4. *For any odd integer $m > 0$,*

$$I\!I_n^{(m)}(u_0, qu_0, \dots, u_{m+2}, qu_{m+2}; q^2; p, q^2) = I\!I_n^{((m-1)/2)}(u_0, \dots, u_{m+2}, \pm q^{1/2}; q; p, q), \tag{5.24}$$

subject to the balancing condition $q^{2n-1} \prod_{0\leq r<m+3} u_r = -(pq)^{(m+1)/2}$.

Remark. When $m = 1$, this is the aforementioned evaluation. One can relax the condition that m is odd by taking $u_{m+2} = p^{1/2}q^{1/2}$ or $-p^{1/2}q^{1/2}$, thus causing a pair of parameters to cancel on the left, but not the right; similarly, one can change the sign of the balancing condition at the cost of increasing the order on the right.

A similar argument gives the following, univariate only, transformation.

Proposition 5.5. *For any even integer $m \geq 0$,*

$$I^{(m)}(\pm\sqrt{u_0}, \dots, \pm\sqrt{u_{m+2}}; p, q) = 2\Gamma_{p,q}(-1) I^{(m/2)}(u_0, \dots, u_{m+2}, -1, -q, -p; p^2, q^2), \tag{5.25}$$

subject to the balancing condition $\prod_{0\leq r<m+3} u_r = -(pq)^{m+1}$.

Since Conjecture 1 involves a choice of 4-torsion point on the elliptic curves (namely $\sqrt{-1}$), it has an equivalent form under modular transformation. This should then extend back to general partition pairs, although we have rather less guidance in this case. Luckily, the argument for $n = 2$ carries over with little change other than replacing Proposition 5.12 with Proposition 5.6. The resulting integral breaks symmetry between u_1, u_2, but this can be restored by adding an additional parameter, as follows.

Conjecture Q2 $(p^{1/2}, t^{-1/2})$**.** *For otherwise generic parameters satisfying $t^n t_0 t_1 t_2 u_0 = p^{1/2} q$, and even $n \ge 0$, one has*

$$\begin{aligned}\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots, p^{1/4} z_i^{\pm 1}, \dots; p^{1/4} t_0, p^{1/4} u_0; t; p, q) \Delta^{(n/2)}(; t_0, t_1, t_2, u_0, \pm\sqrt{t}, \pm\sqrt{qt}, p^{1/2} v, p^{1/2} q/v; t; p^{1/2}, q)\\ = \frac{\Delta^0_{\boldsymbol{\lambda}}(t^{n-1} t_0/u_0 | t^{n-1} p^{1/2} t_0 t_1; t; p, q)}{\Delta^0_{\boldsymbol{\lambda}}(t^{n-1} t_0/u_0 | t^n t_0 t_1; t; p, q)} \prod_{\substack{0 \le i < n\\ 0 \le r < s < 3}} \frac{\Gamma_{p,q}(t^i p^{1/2} t_r t_s)}{\Gamma_{p,q}(t^{i+1} t_r t_s)} \qquad (5.26)\\ \times \int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots, t^{1/2} z_i^{\pm 1}, \dots; t^{1/2} t_0, t^{1/2} u_0; t; p, q) \Delta^{(n/2)}(; t_0, t t_0, t_1, t t_1, t_2, t t_2, u_0, t u_0, p v, p q/v; t^2; p, q).\end{aligned}$$

Remark. The additional parameter has the effect of multiplying each integrand by

$$\prod_{1 \le i \le n/2} \theta_q(v z_i^{\pm 1}). \qquad (5.27)$$

Since these functions span the ($n/2+1$-dimensional) space of $BC_{n/2}$-symmetric q-theta functions of degree 1 [14, Defn. 1], one may replace this factor by an arbitrary such function without affecting the validity of the conjecture. In particular, for $n = 2$, it suffices to verify the conjecture for two values of v, say $v = t_1$, $v = t_2$, which eliminates the extra parameter, and allows the previous argument to apply. The cases $t = 1$, $t = p$, $t = q$ follow as before.

Remark. Again, this (and the remainder of the conjectures we will formulate along these lines) is consistent with respect to connection coefficients, $\boldsymbol{\lambda} \mapsto (l, m)^n + \boldsymbol{\lambda}$, and shifts in t_1, t_2, regardless of the additional parameter. Another important consistency condition is that if we take $v = t_2$ then multiply t_2 by $p^{-1/2}$, the left-hand side is again symmetric in t_1 and t_2, while an application of Conjecture L1 exhibits the corresponding symmetry on the right-hand side.

The corresponding vanishing conjecture (obtained by taking $v = t_2$ to eliminate the extra parameter, then $t_2 = q^{1/2} t^{-1/2}$ or $t_1 = p^{1/2} q^{1/2} t^{-1/2}$, then applying Theorem 4.7) is as follows.

Conjecture 2. *For generic parameters satisfying $t^{2n-1/2} t_0 t_1 u_0 = p^{1/2} q^{1/2}$, the integral*

$$\langle \tilde{\mathcal{R}}^{(2n)}_{\boldsymbol{\lambda}}(\dots, p^{1/4} z_i^{\pm 1}, \dots; p^{1/4} t_0{:}p^{-1/4} t_0, p^{1/4} t_1, p^{-1/4} t_1; p^{1/4} u_0, p^{-1/4} t u_0; t; p, q)\rangle^{(n)}_{t_0, t_1, u_0, t^{1/2}, -t^{1/2}, -q^{1/2} t^{1/2}; t; p^{1/2}, q} \qquad (5.28)$$

vanishes unless $\boldsymbol{\lambda}$ is of the form $\boldsymbol{\mu}^2$, in which case the integral is

$$\frac{\Delta_{\boldsymbol{\mu}}(1/t^2 u_0^2 | t^{2n}, t^{2n-1} t_0^2, 1/t^{2n-1} t_0 u_0, 1/t^{2n} t_0 u_0; t^2; p, q)}{\Delta_{\boldsymbol{\mu}^2}(1/t u_0^2 | t^{2n}, t^{2n-1} t_0^2, 1/t^{2n-1} t_0 u_0, 1/t^{2n} t_0 u_0; t; p, q)}. \qquad (5.29)$$

Remark. Note that the nonzero values are the same as in Conjecture 1.

This would imply Conjecture 1 via a modular transformation, as discussed above, but the q-elliptic half of the identity would be new. In the limit $q \to 0$, t_0, t_1 fixed of that q-elliptic identity, the biorthogonal function becomes a Koornwinder polynomial, and one obtains the vanishing identity given as Theorem 4.10 of [17], together with a conjecture for the nonzero values. The case $t_0 = q^{1/2} t^{1/2}$, $t_1 \mapsto p^{1/4} a$ is also of interest, as in that case the biorthogonal function becomes a suitably normalized interpolation function. One can then take the limit $p \to 0$ with a fixed, in which limit the integral becomes

$$\int \frac{P_\lambda(\dots,z_i^{\pm 1},\dots;q,t)}{P_\lambda(q^{-1/2}t^{1/2-2n},\dots,q^{-1/2}t^{-1/2};q,t)} \prod_{1\le i<j\le n} \frac{(z_i^{\pm 1}z_j^{\pm 1};q)}{(tz_i^{\pm 1}z_j^{\pm 1};q)} \prod_{1\le i\le n} \frac{(z_i^{\pm 2};q)}{(tz_i^{\pm 2};q)} \frac{dz_i}{2\pi\sqrt{-1}z_i}. \tag{5.30}$$

Apart from the change in normalization of the Macdonald polynomial, this integral is that of Theorem 4.1 of [17], and therefore vanishes unless $\lambda = \mu^2$, as predicted by the conjecture. Moreover, the corresponding nonzero values (known in this case) agree with those obtained by degenerating Conjecture 2. We also recall from [13] that in the limit $n \to \infty$ this becomes Macdonald's Littlewood identity.

Again, there is a sum version, this time based on the identity

$$\mathcal{R}^{*(2n)}_{\boldsymbol\lambda}(\dots p^{1/4}z_i^{\pm 1}\dots;p^{3/4}q/t^{2n}u_0,p^{1/4}u_0;t;p,q) = \mathcal{R}^{*(n)}_{(2,1)\boldsymbol\lambda}(\dots z_i\dots;p^{1/2}q/t^n u_0,u_0;t;p^{1/2},q) \tag{5.31}$$

of Cauchy-type interpolation functions (suitably extended); we omit the details. When $\ell(\boldsymbol\lambda) \le 1$, the sum is a special case of the discrete version of Proposition 5.6 below (which discrete version in turn combines a quadratic transform of Warnaar [27] with the modular transform of the transform of Spiridonov [22, Thm. 5.1] mentioned above).

Of course, the next step is to dualize the above conjectures; however, we see by reference to the known trigonometric cases that some subtleties will arise. The vanishing integral of Theorem 4.8 of [17] (corresponding to Conjecture 1) actually dualizes to a pair of vanishing integrals (depending on whether the number of variables is even or odd), while for Theorem 4.1 of [17] (a limit of Conjecture 2), not only are there two dual identities, but each dual identity itself involves a sum of two integrals.

One case is straightforward, namely the "other" dual of Conjecture Q2 (i.e., exchange p and q before dualizing). Here, and in the other two cases, we begin by dualizing the algebraic versions of the conjectures, à la (5.8), having first analytically continued in $t^{2n} = T$. After reparametrizing and specializing T appropriately, we can recognize the left-hand side as the integral of a Cauchy-type interpolation function, and thus reexpress the dual as a vanishing identity. At that point, one may use the Pieri identity to extend to a large set of cases of an integral transformation.

Conjecture Q3 $(t^{-1/2},q^{1/2})$**.** *Subject to the balancing conditions $t^{n-1}t_0t_1t_2u_0 = pq^2t^{1/2}$, $tv_0v_1 = pq$, one has*

$$\begin{aligned}\int &\mathcal{R}^{*(n)}_{\boldsymbol\lambda}(;t^{-1/4}t_0,t^{-1/4}u_0;t;p,q)\\ &\times \Delta^{(n)}(;t^{-1/4}t_0,t^{-1/4}t_1,t^{-1/4}t_2,t^{-1/4}u_0,\pm t^{1/4},\pm p^{1/2}t^{1/4},t^{1/4}v_0,t^{1/4}v_1;t^{1/2};p,q)\\ &\quad= \frac{\Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0/u_0|t^{n-3/2}t_0t_1;t;p,q)}{\Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0/u_0|t^{n-1}t_0t_1/q;t;p,q)} \prod_{0\le i<n}\prod_{0\le r<s<3} \frac{\Gamma_{p,q}(t^{i-1/2}t_rt_s)}{\Gamma_{p,q}(t^it_rt_s/q)}\\ &\qquad\times \int \mathcal{R}^{*(n)}_{\boldsymbol\lambda}(;q^{-1/2}t_0,q^{-1/2}u_0;t;p,q)\Delta^{(n)}(;q^{\pm 1/2}t_0,q^{\pm 1/2}t_1,q^{\pm 1/2}t_2,q^{\pm 1/2}u_0,q^{1/2}v_0,q^{1/2}v_1;t;p,q^2)\end{aligned} \tag{5.32}$$

Remark. Here the extra parameter multiplies the integrands by factors

$$\prod_{0\le i<n} \frac{\Gamma_{p,q}(t^{1/4}v_0z_i^{\pm 1})}{\Gamma_{p,q}(t^{3/4}v_0z_i^{\pm 1})} \quad\text{and}\quad \prod_{0\le i<n} \frac{\Gamma_{p,q^2}(q^{1/2}v_0z_i^{\pm 1})}{\Gamma_{p,q^2}(q^{1/2}tv_0z_i^{\pm 1})}, \tag{5.33}$$

which are not, in fact, theta functions. They do, however, closely resemble the generating function for the q,t-analogues g_k of the complete symmetric functions [10].

Remark. If we eliminate the extra parameters from the normalization (i.e., $\boldsymbol{\lambda}=0$) cases of this conjecture and Conjecture Q2, the resulting quadratic transformations are equivalent by Proposition 1.1.

Proposition 5.6. *Conjecture Q3 holds when $n=1$.*

Proof. Since $\ell(\boldsymbol{\lambda})\le 1$, we immediately reduce to the case $\boldsymbol{\lambda}=0$, for which we need simply exchange order of integration in the double integral (which on an open set of parameters can use unit circle contours)

$$\int\int \Gamma_{p,q}(q^{-1/2}t^{1/4}x^{\pm 1}y^{\pm 1})\frac{\prod_{0\le r<4}\Gamma_{p,q}(t^{-1/4}u_r x^{\pm 1})}{\Gamma_{p,q}(x^{\pm 2})}\frac{dx}{2\pi\sqrt{-1}x}\frac{\Gamma_{p,q^2}(q^{1/2}v_0 y^{\pm 1},q^{1/2}v_1 y^{\pm 1})}{\Gamma_{p,q^2}(y^{\pm 2})}\frac{dy}{2\pi\sqrt{-1}y} \tag{5.34}$$

and simplify the resulting integrands. ∎

Unfortunately, this case is not sufficient to prove the $t=p$ and $t=q$ cases, as although they can again (and for the later conjectures) be expressed via (generalized) pfaffians, the entries of the pfaffians include instances with $n=2$. The univariate case does, however, suffice to prove the case $t=1$.

Regarding the $n=2$ case, we note that it suffices to prove the case $\boldsymbol{\lambda}=0$; indeed, the analogue of the argument in Proposition 5.3 applies, although the functions coming from parameter shifts only span for $n\le 3$. (In fact, Conjecture Q3 has recently been proved for $\boldsymbol{\lambda}=0$ by Van de Bult [4], thus implying the general $n\le 3$ case as well as the general pfaffian cases.)

The corresponding vanishing conjecture states that

$$\langle \tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\lambda}}(;t^{-1/4}t_0{:}t^{1/4}t_0,t^{-1/4}t_1,t^{1/4}t_1;t^{-1/4}u_0,t^{1/4}u_0/q;t;p,q)\rangle^{(n)}_{t^{-1/4}t_0,t^{-1/4}t_1,t^{-1/4}u_0,t^{1/4},-t^{1/4},-p^{1/2}t^{1/4};t^{1/2};p,q} \tag{5.35}$$

vanishes unless $\boldsymbol{\lambda}=(1,2)\boldsymbol{\mu}$, when it equals

$$\frac{\Delta_{\boldsymbol{\mu}}(q/u_0^2|t^n,t^{n-1}t_0^2,1/t^{n-1}t_0u_0,q/t^{n-1}t_0u_0;t;p,q^2)}{\Delta_{(1,2)\boldsymbol{\mu}}(q/u_0^2|t^n,t^{n-1}t_0^2,1/t^{n-1}t_0u_0,q/t^{n-1}t_0u_0;t;p,q)}. \tag{5.36}$$

The Koornwinder-type limit $p\to 0$ is again a vanishing integral of [17] (the dual of Theorem 4.10 op. cit.), together with a conjecture for the nonzero values.

The next simplest case is the dual of Conjecture Q1. Here we find that the integral on the left-hand side is half the dimension of that on the right-hand side, which is thus necessarily even. This constraint can be avoided, however, by observing that the corresponding integral of Cauchy-type interpolation functions, when analytically continued in $T=t^{2n}$, then specialized to $T=t^{2n+1}$ can still be expressed as an integral. One thus obtains the following conjecture.

Conjecture Q4 $(-1,q^{1/2})$**.** *For otherwise generic parameters satisfying $t^{n-1}t_0t_1t_2u_0=-pq^2$, the integral*

$$\frac{\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0/u_0|-t^{n-1}t_0t_1;t;p,q)}{\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0/u_0|t^{n-1}t_0t_1/q;t;p,q)}\prod_{0\le i<n}\prod_{0\le r<s<3}\frac{\Gamma_{p,q}(-t^i t_r t_s)}{\Gamma_{p,q}(t^i t_r t_s/q)}$$
$$\times\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots z_i\dots;q^{-1/2}t_0,q^{-1/2}u_0;t;p,q)\Delta^{(n)}(;q^{\pm 1/2}t_0,q^{\pm 1/2}t_1,q^{\pm 1/2}t_2,q^{\pm 1/2}u_0;t;p,q^2) \tag{5.37}$$

is equal to

$$\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,\pm\sqrt{-z_i},\dots;t_0\sqrt{-1},u_0\sqrt{-1};t;p,q)\Delta^{(n/2)}(;t_0^2,t_1^2,t_2^2,u_0^2,1,p,t,pt;t^2;p^2,q^2) \tag{5.38}$$

if n is even, and

$$\begin{aligned}&\Gamma_{p^2,q^2}(t_0^2,t_1^2,t_2^2,u_0^2,p,t,pt)\\ &\times \int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,\pm\sqrt{-z_i},\dots,\sqrt{-1};t_0\sqrt{-1},u_0\sqrt{-1};t;p,q)\Delta^{((n-1)/2)}(;t_0^2,t_1^2,t_2^2,u_0^2,t^2,p,t,pt;t^2;p^2,q^2)\end{aligned} \tag{5.39}$$

if n is odd.

Again, this is consistent with respect to parameter shifts and permutations and annihilates the kernel of the relevant homomorphism; the latter involves a (formal) special case of the adjointness arguments in the proof of Proposition 4.13.

Proposition 5.7. *If $t=q^2$, then Conjecture Q4 holds if $\boldsymbol{\lambda}=0$ or $n\le 7$.*

Proof. As in Proposition 5.3, one side is an elliptic Selberg integral, while the other can be evaluated using Proposition 5.4. The extension to general $\boldsymbol{\lambda}$ when $n\le 7$ is similar as well. $\square$

In particular, when $n=1$, both sides are essentially independent of t, and thus the conjecture holds in that case as well. Note, however, that the usual deduction of the $t=1$ case from the univariate case founders on the fact that the integral becomes singular when $t=1$. The cases $t\in\{p,q\}$ follow from the fact that the conjecture holds for $n=1$ and $n=2$.

Theorem 5.8. *Conjecture Q4 holds for $n\le 3$.*

Proof. As before, we may reduce to the case $\boldsymbol{\lambda}=0$. Let $F(t_0,t_1,t_2)$ denote the right-hand side of the conjecture, either

$$I\!I^{(1)}_1(t_0^2,t_1^2,t_2^2,u_0^2,1,p,t,pt;t^2;p^2,q^2) \tag{5.40}$$

or

$$\Gamma_{p^2,q^2}(t_0^2,t_1^2,t_2^2,u_0^2,p,t,pq)I\!I^{(1)}_1(t_0^2,t_1^2,t_2^2,u_0^2,t^2,p,t,pt;t^2;p^2,q^2), \tag{5.41}$$

depending on whether $n=2$ or $n=3$; we solve for u_0 via the balancing condition. Similarly, let $G(t_0,t_1,t_2)$ denote the corresponding left-hand side.

Now, it follows as a special case of the general elliptic hypergeometric equation [23] (also Spiridonov's habilitation thesis) that F satisfies a pair of difference equations

$$F(qt_0,t_1,t_2)=A(t_0,t_1,t_2)F(t_0,t_1,t_2)+B(t_0,t_1,t_2)F(t_0/q,t_1,t_2) \tag{5.42}$$

$$F(pt_0,t_1,t_2)=C(t_0,t_1,t_2)F(t_0,t_1,t_2)+D(t_0,t_1,t_2)F(t_0/p,t_1,t_2), \tag{5.43}$$

where A and B are p-theta functions and C and D are q-theta functions, the specific formulas for which we will not use. (Moreover, there exists a rescaling, see [16], that makes the coefficients elliptic functions of t_0.) By Corollary 11 of [16], if either of these equations has generically irreducible Galois group, then F is the unique solution of the pair of equations, up to a factor independent of t_0. Since irreducibility is preserved under degeneration [1], we may take a limit $q \to 1$ with $t \to 1$, $t_0 \to \sqrt{-p}$, t_2, $t_2 \to \sqrt{-1}$. The result depends on the various rates of approach, and is an Euler integral evaluated at $s = \lambda(p)$, where λ is the cross-ratio of the 2-torsion of the elliptic curve with modular parameter p, and the only other constraint is that two of the exponents are equal. Thus in particular we can obtain a general complete elliptic integral as a limit, and since the corresponding equation is a second-order differential equation with nonelementary solutions, it has irreducible monodromy. Since both sides agree when $t_0 = p^{1/2}q$, we conclude that it indeed suffices to show that G satisfies the same equations.

Following Spiridonov, the first stage in deriving the elliptic hypergeometric equation is to observe that the integrands of the three integrals

$$F(t_0, t_1, t_2), F(qt_0, t_1, t_2), F(t_0, qt_1, t_2) \tag{5.44}$$

are linearly dependent. Now, by consistency of the conjecture with respect to parameter shifts, we can write each integrand as the $F(t_0, t_1, t_2)$ integrand times a function $f(\pm\sqrt{-z}, \{\sqrt{-1}\})$ in such a way that expanding f in interpolation functions and applying the conjecture term-by-term gives the corresponding shift of G. But consistency with respect to the homomorphism tells us that any linear combination of such functions that makes the image vanish makes the transformed integral vanish as well. It follows that the three integrals

$$G(t_0, t_1, t_2), G(qt_0, t_1, t_2), G(t_0, qt_1, t_2) \tag{5.45}$$

satisfy the same dependence.

The next step in Spiridonov's derivation is to use an E_7 transformation to obtain relations between the integrals

$$F(t_0, t_1, t_2), F(t_0/q, t_1, t_2), F(t_0, t_1/q, t_2). \tag{5.46}$$

It turns out that for a suitable choice of element (different from Spiridonov's), we can do so with transformations that preserves the forms of the integrals. We find, in particular, that

$$F(t_0, t_1, t_2) = \prod_{0 \le i < n} \Gamma_{p,q^2}(t^i t_0^2, t^i t_1^2, t^i t_2^2, t^i u_0^2) F(\sqrt{pq^2/t^{n-1}}/t_0, \sqrt{pq^2/t^{n-1}}/t_1, \sqrt{pq^2/t^{n-1}}/t_2), \tag{5.47}$$

and similarly for G. (For F we transform using equation (9.49) of [15] (the composition of the reflection in $(1/2, 1/2, 1/2, 1/2, -1/2, -1/2, -1/2, -1/2)$ with the central element of $W(E_7)$), while for G we use Corollary 9.13 of [15] (the central element).) But this reverses the direction of shifting, as required.

These two recurrences are enough to generate the difference equation: using the second recurrence with $t_1 \mapsto qt_1$, we can express $F(t_0, t_1, t_2)$ in terms of $F(t_0, qt_1, t_2)$ and $F(t_0/q, qt_1, t_2)$, and these can in turn be expressed using the first recurrence in terms of $F(qt_0, t_1, t_2)$, $F(t_0, t_1, t_2)$ and $F(t_0/q, t_1, t_2)$. ☐

Remark. Similar considerations also produce recurrences for $n = 4$ and $n = 5$, but these do not appear to be enough to generate a difference equation.

The corresponding vanishing conjecture (take $t_2 = p^{1/2}q$) reads that the integrals

$$\langle \tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\lambda}}(\dots, \pm\sqrt{-z_i}, \dots; t_0\sqrt{-1}{:} -t_0\sqrt{-1}, \pm t_1\sqrt{-1}; u_0\sqrt{-1}, -u_0\sqrt{-1}/q; t; p, q)\rangle^{(n/2)}_{t_0^2, u_0^2, t_1^2, 1, t, pt; t^2; p^2, q^2} \tag{5.48}$$

and

$$\langle \tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\lambda}}(\dots,\pm\sqrt{-z_i},\dots,\sqrt{-1};t_0\sqrt{-1}{:}-t_0\sqrt{-1},\pm t_1\sqrt{-1};u_0\sqrt{-1},-u_0\sqrt{-1}/q;t;p,q)\rangle^{((n-1)/2)}_{t_0^2,u_0^2,t_1^2,t^2,t,pt;t^2;p^2,q^2}, \tag{5.49}$$

subject to the balancing condition $t^{2n-2}t_0^2t_1^2u_0^2 = pq^2$, vanish unless $\boldsymbol{\lambda}$ has the form $(1,2)\boldsymbol{\mu}$, when the value is as in (5.36) above. The $n=1$ instance of this is a quadratic evaluation formula due to Warnaar [28, (1.4,1.10)].

The remaining case of the three is the direct dual of Conjecture 2. If we attempt to proceed as above, we find that the most straightforward version of the half-integer case fails to hold; although taking $t_1 = p^{1/2}q$ reduces to the dual vanishing identity, taking $t_1 = -p^{1/2}q$ makes the right-hand side vanish. If one instead takes a sum of two terms, symmetric under $p^{1/2}\mapsto -p^{1/2}$, this problem disappears. This, of course, corresponds to the fact that the known Macdonald polynomial limit itself involves a sum of two integrals. The corresponding structure for the integer case is then reasonably straightforward to guess. One thus formulates the following conjecture.

Conjecture Q5 $(p^{1/2},q^{1/2})$**.** *For otherwise generic parameters satisfying $t^{n-1}t_0t_1t_2u_0 = p^{1/2}q^2$, the rescaled integral*

$$\frac{\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0/u_0|t^{n-1}p^{1/2}t_0t_1;t;p,q)}{\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0/u_0|t^{n-1}t_0t_1/q;t;p,q)}\prod_{\substack{0\le i<n\\ 0\le r<s<3}}\frac{\Gamma_{p,q}(t^ip^{1/2}t_rt_s)}{\Gamma_{p,q}(t^it_rt_s/q)}$$
$$\times\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(;q^{-1/2}t_0,q^{-1/2}u_0;t;p,q)\Delta^{(n)}(;q^{\pm 1/2}t_0,q^{\pm 1/2}t_1,q^{\pm 1/2}t_2,q^{\pm 1/2}u_0,pqv^{\pm 1};t;p,q^2) \tag{5.50}$$

admits the following expressions as sums of lower-dimensional integrals:

If n is odd, then it equals

$$\begin{aligned}
&\Gamma_{p^{1/2},q}(t_0,t_1,t_2,u_0,-1,\pm t^{1/2},p^{1/2}q^{1/2}v^{\pm 1})\\
&\qquad\times\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,p^{1/4}z_i^{\pm 1},\dots,p^{1/4};p^{1/4}t_0,p^{1/4}u_0;t;p,q)\\
&\qquad\qquad\times\Delta^{((n-1)/2)}(;t_0,t_1,t_2,u_0,t,-1,\pm t^{1/2},p^{1/2}q^{1/2}v^{\pm 1};t;p^{1/2},q)\\
&+\Gamma_{p^{1/2},q}(-t_0,-t_1,-t_2,-u_0,-1,\pm t^{1/2},-p^{1/2}q^{1/2}v^{\pm 1})\\
&\qquad\times\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,p^{1/4}z_i^{\pm 1},\dots,-p^{1/4};p^{1/4}t_0,p^{1/4}u_0;t;p,q)\\
&\qquad\qquad\times\Delta^{((n-1)/2)}(;t_0,t_1,t_2,u_0,1,-t,\pm t^{1/2},p^{1/2}q^{1/2}v^{\pm 1};t;p^{1/2},q),
\end{aligned} \tag{5.51}$$

while if n is even, it equals

$$\begin{aligned}
&\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,p^{1/4}z_i^{\pm 1},\dots;p^{1/4}t_0,p^{1/4}u_0;t;p,q)\\
&\qquad\times\Delta^{(n/2)}(;t_0,t_1,t_2,u_0,\pm 1,\pm t^{1/2},p^{1/2}q^{1/2}v^{\pm 1};t;p^{1/2},q)\\
&+\Gamma_{p,q^2}(t_0^2,t_1^2,t_2^2,u_0^2,t,t,pqv^{\pm 2})\Gamma_{p^{1/2},q}(-1,-t)\\
&\qquad\times\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,p^{1/4}z_i^{\pm 1},\dots,\pm p^{1/4};p^{1/4}t_0,p^{1/4}u_0;t;p,q)\\
&\qquad\qquad\times\Delta^{(n/2-1)}(;t_0,t_1,t_2,u_0,\pm t,\pm t^{1/2},p^{1/2}q^{1/2}v^{\pm 1};t;p^{1/2},q).
\end{aligned} \tag{5.52}$$

Remark. Here the extra parameter multiplies the first integrand by

$$g(z_1,\dots,z_n) = \prod_{1\le i\le n} \theta_{q^2}(qvz_i^{\pm 1}), \tag{5.53}$$

while the integrands on the right are multiplied by

$$g(q^{1/2}z_1^{\pm 1},\dots,q^{1/2}z_{(n-1)/2}^{\pm 1},q^{1/2}), \qquad g(q^{1/2}z_1^{\pm 1},\dots,q^{1/2}z_{(n-1)/2}^{\pm 1},-q^{1/2}), \tag{5.54}$$

$$g(q^{1/2}z_1^{\pm 1},\dots,q^{1/2}z_{n/2}^{\pm 1}), \qquad g(q^{1/2}z_1^{\pm 1},\dots,q^{1/2}z_{n/2-1}^{\pm 1},\pm q^{1/2}), \tag{5.55}$$

and this factor accounts for all dependence on v. One could thus just as well replace g by an arbitrary BC_n-symmetric q^2-theta function of degree 1. From this perspective, each of the above four specializations induces a linear transformation from the space of such theta functions to a corresponding space of BC_n-symmetric q-theta functions. Moreover, each such transformation is surjective, with kernel one of the two eigenspaces of the operator

$$g(z_1,\dots,z_n) \to q^{n/2}(\prod_{1\le i\le n} z_i)g(qz_1,\dots,qz_n). \tag{5.56}$$

The interpolation functions are similarly specialized, and thus if g is an eigenfunction, then we should expect the integral to vanish when the interpolation function on the right is replaced by anything in the corresponding kernel. The q-elliptic part of this kernel is not an eigenspace of an involution, so the earlier argument fails on that half; we thus only consider the case that the p-elliptic portion of the integrand is in the kernel. We can then argue as we did after Conjecture Q4, to find that the corresponding left-hand side can again be expressed in terms of a difference operator, and thus reduce to showing that the integral

$$\begin{aligned}\int \mathcal{D}_q^{+(n)}&(q^{-1/2}u_0{:}q^{-1/2}t_0,q^{-1/2}t_1,q^{-1/2}t_2,q^{1/2}p^{3/4}w;t;p)f\\ &\times\Delta^{(n)}(;q^{\pm 1/2}t_0,q^{\pm 1/2}t_1,q^{\pm 1/2}t_2,q^{\pm 1/2}u_0,pqv^{\pm 1};t;p,q^2)\end{aligned} \tag{5.57}$$

is quasiperiodic (multiplied by $(q^{1/2}p^{1/4}vw)^{-n}$) under $(v,w)\mapsto(qv,p^{1/2}w)$. But this again follows by an adjointness argument.

Proposition 5.9. *Conjecture Q5 holds when $n=1$.*

Proof. Since $\ell(\boldsymbol{\lambda})\le 1$ when $n=1$, we may as well take $\boldsymbol{\lambda}=0$. We thus need simply to prove that when $u_0u_1u_2u_3=p^{1/2}q^2$,

$$\begin{aligned}&\frac{(p;p)(q^2;q^2)}{2}\int \frac{\prod_{0\le r<4}\Gamma_{p,q^2}(q^{\pm 1/2}u_rz^{\pm 1})}{\Gamma_{p,q^2}(z^{\pm 2})}\frac{\theta_{q^2}(qvz^{\pm 1})dz}{2\pi\sqrt{-1}z}\\ &\quad=\frac{\Gamma_{p^{1/2},q}(-1,u_0,u_1,u_2,u_3)\theta_q(q^{1/2}v)}{\prod_{0\le r<s<4}\Gamma_{p,q}(p^{1/2}u_ru_s)}+\frac{\Gamma_{p^{1/2},q}(-1,-u_0,-u_1,-u_2,-u_3)\theta_q(-q^{1/2}v)}{\prod_{0\le r<s<4}\Gamma_{p,q}(p^{1/2}u_ru_s)}\end{aligned} \tag{5.58}$$

Each of the three terms is a BC_1-symmetric q^2-theta function in v of degree 1, and thus the relation will follow if we check it at any two independent points. By symmetry under $v\mapsto -v$, we may reduce to the case $v=q^{-1/2}$, when the left-hand side can be expressed (via Proposition 5.4) as

$$\frac{(q^2;q^2)}{(q;q)}I^{(0)}(q^{-1/2}u_0,q^{-1/2}u_1,q^{-1/2}u_2,q^{-1/2}u_3,-q^{1/2},-p^{1/2}q^{1/2};p,q). \tag{5.59}$$

The proposition follows upon simplifying the resulting product of elliptic gamma functions. ☐

Remark. This can also be obtained as the limit $t \to 1/p$ of Proposition 5.6; one finds that the left-hand side of that Proposition violates the contour conditions in two different ways in the limit, and thus becomes a sum of two residues, corresponding to the two terms above.

The proof of Theorem 5.8 carries over, with some additional subtleties.

Theorem 5.10. *Conjecture Q5 holds whenever $n \le 3$.*

Proof. As before, we may reduce to the case $\boldsymbol{\lambda} = 0$. In addition, the above considerations involving the function g have the effect that the two terms on the right-hand side are q-theta functions of v, but with different multipliers. We may thus use this to write either term on the right as a linear combination of two instances of the term on the left. Finally, it suffices to consider the case $v = q^{-1/2}u_0$, since the $\boldsymbol{\lambda} = 0$ case is symmetrical between u_0 and the t_r parameters. We can then argue as in Theorem 5.8 to see that both sides satisfy the same elliptic q-difference equations. Since we only have consistency with respect to the p-elliptic kernel, this does not give us the requisite p-difference equation to finish the proof. However, if we shift $u_0 \mapsto p^{-1/2}u_0$ and square p, the integrals on the right become symmetrical under permutations of t_0, t_1, t_2, u_0 as well as under swapping p and q. For $n = 2$, the desired identity becomes

$$\begin{aligned}\prod_{0\le r<s<3} \frac{\Gamma_{p^2,q}(pt_rt_s, ptt_rt_s)}{\Gamma_{p^2,q}(t_rt_s/q, tt_rt_s/q)} &I\!I_2^{(1)}(q^{\pm 1/2}t_0, q^{\pm 1/2}t_1, q^{\pm 1/2}t_2, (p^2q)^{\pm 1/2}u_0; t; p^2, q^2) \\ &= I\!I_1^{(1)}(t_0, t_1, t_2, u_0, \pm 1, \pm t^{1/2}; t; p, q) + \Gamma_{p^2,q^2}(t_0^2, t_1^2, t_2^2, u_0^2, t, t)\Gamma_{p,q}(-1, -t), \end{aligned} \tag{5.60}$$

(with balancing condition $tt_0t_1t_2u_0 = p^2q^2$), while for $n = 3$, it becomes

$$\begin{aligned}\prod_{\substack{0\le i<3\\0\le r<s<3}} \frac{\Gamma_{p^2,q}(t^ipt_rt_s)}{\Gamma_{p^2,q}(t^it_rt_s/q)} &I\!I_3^{(1)}(q^{\pm 1/2}t_0, q^{\pm 1/2}t_1, q^{\pm 1/2}t_2, (p^2q)^{\pm 1/2}u_0; t; p^2, q^2) \\ = \quad &\Gamma_{p,q}(t_0, t_1, t_2, u_0, -1, \pm t^{1/2}) I\!I_1^{(1)}(t_0, t_1, t_2, u_0, t, -1, \pm t^{1/2}; t; p, q) \\ &+ \Gamma_{p,q}(-t_0, -t_1, -t_2, -u_0, -1, \pm t^{1/2}) I\!I_1^{(1)}(t_0, t_1, t_2, u_0, 1, -t, \pm t^{1/2}; t; p, q), \end{aligned} \tag{5.61}$$

with balancing condition $t^2t_0t_1t_2u_0 = p^2q^2$. The corresponding symmetries of the left-hand side follow from Proposition 1.1, as do the relevant symmetries when $v = q^{-3/2}u_0$. Thus, the fact that both sides satisfy the same p-difference equations implies that both sides satisfy the same q-difference equations, and the identity follows. ∎

Remark. The extension to $t = p$, $t = q$ can also be made to work; if one represents the left-hand side as a "pfaffian" of $n = 2$ and $n = 1$ instances, then the transform is a sum of $2^{\lceil n/2 \rceil}$ "pfaffians". Most of these vanish, however, and the survivors are all proportional to one of the two terms of the right-hand side. (When $\lambda = 0$ so the "pfaffian" is actually a pfaffian, the corresponding alternating matrix is a sum of two alternating matrices, one of rank 2.)

When $t_1 = -p^{1/2}q$, $v = q^{1/2}/t_2$, one of the two terms vanishes, since $\Gamma(p^{1/2}q; p^{1/2}, q) = 0$, and we thus obtain (after an application of Conjecture L2) the following vanishing conjecture: that when $t^{n-1}t_0t_1u_0 = -p^{1/2}q$, the integral

$$\langle \tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\lambda}}(\dots, p^{1/4}z_i^{\pm 1}, \dots; p^{1/4}t_0{:}p^{-1/4}t_0, p^{1/4}t_1, p^{-1/4}t_1; p^{1/4}u_0, p^{-1/4}u_0/q; t; p, q)\rangle^{(n/2)}_{t_0,t_1,u_0,1,t^{1/2},-t^{1/2};t;p^{1/2},q} \tag{5.62}$$

or

$$\langle\tilde{\mathcal{R}}^{(n)}_{\boldsymbol{\lambda}}(\dots,p^{1/4}z_i^{\pm 1},\dots,p^{1/4};p^{1/4}t_0{:}p^{-1/4}t_0,p^{1/4}t_1,p^{-1/4}t_1;p^{1/4}u_0,p^{-1/4}u_0/q;t;p,q)\rangle^{((n-1)/2)}_{t_0,t_1,u_0,t,t^{1/2},-t^{1/2};t;p^{1/2},q}, \tag{5.63}$$

as appropriate, vanishes unless $\boldsymbol{\lambda}=(1,2)\boldsymbol{\mu}$, when its value is given by (5.36).

Similarly, in the limit $t_1,t_2\to q^{1/2}v^{\pm 1}$, the n-dimensional integral degenerates to a (dual) Littlewood-style sum, and an application of connection coefficients gives the conjecture that when $t^{n-1}t_0u_0=p^{1/2}q$, the integrals

$$\begin{aligned}&\frac{1}{2}\langle\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,p^{1/4}z_i^{\pm 1},\dots;p^{-1/4}t_0,p^{1/4}u_0;t;p,q)\rangle^{(n/2)}_{t_0,u_0,1,-1,t^{1/2},-t^{1/2};t;p^{1/2},q}\\&+\frac{1}{2}\langle\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,p^{1/4}z_i^{\pm 1},\dots,\pm p^{1/4};p^{-1/4}t_0,p^{1/4}u_0;t;p,q)\rangle^{(n/2-1)}_{t_0,u_0,t,-t,t^{1/2},-t^{1/2};t;p^{1/2},q}\end{aligned} \tag{5.64}$$

and

$$\begin{aligned}&\frac{1}{2}\langle\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,p^{1/4}z_i^{\pm 1},\dots,p^{1/4};p^{-1/4}t_0,p^{1/4}u_0;t;p,q)\rangle^{((n-1)/2)}_{t_0,u_0,t,-1,t^{1/2},-t^{1/2};t;p^{1/2},q}\\&+\frac{1}{2}\langle\mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,p^{1/4}z_i^{\pm 1},\dots,-p^{1/4};p^{-1/4}t_0,p^{1/4}u_0;t;p,q)\rangle^{((n-1)/2)}_{t_0,u_0,1,-t,t^{1/2},-t^{1/2};t;p^{1/2},q}\end{aligned} \tag{5.65}$$

vanish unless $\boldsymbol{\lambda}=(1,2)\boldsymbol{\mu}$, when they have value

$$\frac{\Delta_{\boldsymbol{\mu}}(q/u_0^2|t^n,t^{n-1};t;p,q^2)}{\Delta_{(1,2)\boldsymbol{\mu}}(q/u_0^2|t^n,t^{n-1};t;p,q)}. \tag{5.66}$$

Taking $t_0=p^{1/4}a$ and $p\to 0$ turns the interpolation functions into Macdonald polynomials, and one again obtains a result of [17] (the dual of Theorem 4.1 op. cit.). When $q=t$, the Macdonald polynomials become Schur functions, and one obtains the well-known representation-theoretic fact that the Haar integral

$$\int_{O\in O(n)} s_\lambda(O) \tag{5.67}$$

vanishes unless $\lambda=2\mu$, when it equals 1.

Remark. One might think to obtain the interpolation function case by a limit of the biorthogonal function case (as this works in the other cases). However, the interpolation function limit only works when the parameters are otherwise generic; in this instance it fails when $\ell(\boldsymbol{\lambda})=n$, as the biorthogonal function becomes singular (the first two parameters multiply to 1).

If we swap p and q above, we find that Conjecture Q4 becomes self-dual, while Conjectures Q3 and Q5 become dual to each other. However, we now have the possibility again of modular transformations. Given the lack of guidance from the trigonometric level, the resulting conjectures are rather more speculative than those above. The overall form of the integrals is fairly straightforward to determine, especially since in each case the normalization without extra parameter reduces via Proposition 1.1 to a previously conjectured normalization. The $\boldsymbol{\lambda}$-dependent factors are then uniquely determined by the requirement of consistency under the Pieri identity (more precisely, that the obvious argument for consistency should work, as it did in all previous cases).

For Conjecture Q4, one obtains the following transform, of which the case $n=1$ is straightforward.

Conjecture Q6 $(p^{1/2},-1)$**.** *For otherwise generic parameters satisfying $t^{n-1}t_0t_1t_2u_0 = p^{1/2}q$, the integral*

$$\frac{\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0/u_0|p^{1/2}t^{n-1}t_0t_1;t;p,q)}{\Delta^0_{\boldsymbol{\lambda}}(t^{n-1}t_0/u_0|t^{n-1}t_0t_1;t;p,q)} \prod_{0\le i<n}\prod_{0\le r<s<3} \frac{\Gamma_{p,q}(t^ip^{1/2}t_rt_s)}{\Gamma_{p,q}(t^it_rt_s)}$$
$$\times \int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,z_i^2,\dots;-t_0,-u_0;t;p,q)\Delta^{(n)}(;\pm\sqrt{-t_0},\pm\sqrt{-t_1},\pm\sqrt{-t_2},\pm\sqrt{-u_0},p^{1/2}q^{1/4}v^{\pm 1};t^{1/2};p^{1/2},q^{1/2}) \tag{5.68}$$

is equal to

$$\int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,p^{1/4}z_i^{\pm 1},\dots;p^{1/4}t_0,p^{1/4}u_0;t;p,q)\Delta^{(n/2)}(;t_0,t_1,t_2,u_0,1,q^{1/2},t^{1/2},q^{1/2}t^{1/2},-p^{1/2}q^{1/2}v^{\pm 2};t;p^{1/2},q) \tag{5.69}$$

if n is even, and

$$\Gamma_{p^{1/2},q}(t_0,t_1,t_2,u_0,q^{1/2},t^{1/2},q^{1/2}t^{1/2},-p^{1/2}q^{1/2}v^{\pm 2})$$
$$\times \int \mathcal{R}^{*(n)}_{\boldsymbol{\lambda}}(\dots,p^{1/4}z_i^{\pm 1},\dots,p^{1/4};p^{1/4}t_0,p^{1/4}u_0;t;p,q)$$
$$\times \Delta^{((n-1)/2)}(;t_0,t_1,t_2,u_0,t,q^{1/2},t^{1/2},q^{1/2}t^{1/2},-p^{1/2}q^{1/2}v^{\pm 2};t;p^{1/2},q) \tag{5.70}$$

if n is odd.

Remark. Once again, the factor

$$\prod_{1\le i\le n}\Gamma_{p^{1/2},q^{!/2}}(p^{1/2}q^{1/4}v^{\pm 1}z_i^{\pm 1}) = \prod_{1\le i\le n}\theta_{q^{1/2}}(q^{1/4}vz_i^{\pm 1}) \tag{5.71}$$

on the left can be replaced by an arbitrary BC_n $q^{1/2}$-theta function g of degree 1, in which case the extra factor on the right is

$$g(\pm\sqrt{-z_1},\dots,\pm\sqrt{-z_{n/2}}),\quad \text{or}\quad g(\pm\sqrt{-z_1},\dots,\pm\sqrt{-z_{(n-1)/2}},\sqrt{-1}), \tag{5.72}$$

as appropriate. The factor on the right vanishes iff

$$g(-z_1,\dots,-z_n) = -g(z_1,\dots,z_n). \tag{5.73}$$

Remark. In fact, algebraically speaking, there is another modular transformation, since this conjecture depends on an ordered pair of 2-torsion points (namely -1 and $p^{1/2}$), so there is a "vanishing" conjecture associated to the pair $(\pm p^{1/2})$. However, this conjecture does not appear amenable to extension to a full integral.

Theorem 5.11. *Conjecture Q6 holds for $n\le 3$.*

Proof. As usual, we may reduce to the case $\boldsymbol{\lambda}=0$, and it will suffice to consider the case $v = q^{-1/4}\sqrt{-u_0}$. If we rescale $u_0\mapsto p^{-1/2}u_0$, we find that we need to prove the identities

$$\prod_{\substack{0\le i<2\\0\le r<s<3}} \frac{\Gamma_{p,q}(t^ip^{1/2}t_rt_s)}{\Gamma_{p,q}(t^it_rt_s)} I\!I_2(\pm\sqrt{-t_0},\pm\sqrt{-t_1},\pm\sqrt{-t_2},p^{1/4}\sqrt{-u_0},-p^{-1/4}\sqrt{-u_0};t^{1/2};p^{1/2},q^{1/2})$$
$$= I\!I_1(t_0,t_1,t_2,u_0,1,q^{1/2},t^{1/2},q^{1/2}t^{1/2};t;p^{1/2},q) \tag{5.74}$$

with $tt_0t_1t_2u_0 = pq$, and

$$\prod_{\substack{0\le i<3\\ 0\le r<s<3}} \frac{\Gamma_{p,q}(t^ip^{1/2}t_rt_s)}{\Gamma_{p,q}(t^it_rt_s)} I\!I_3(\pm\sqrt{-t_0}, \pm\sqrt{-t_1}, \pm\sqrt{-t_2}, p^{1/4}\sqrt{-u_0}, -p^{-1/4}\sqrt{-u_0}; t^{1/2}; p^{1/2}, q^{1/2})$$
$$= \Gamma_{p^{1/2},q}(t_0, t_1, t_2, u_0, q^{1/2}, t^{1/2}, q^{1/2}t^{1/2}) I\!I_1(t_0, t_1, t_2, u_0, t, q^{1/2}, t^{1/2}, q^{1/2}t^{1/2}; t; p^{1/2}, q) \tag{5.75}$$

with $t^2t_0t_1t_2u_0 = pq$. But these identities follow from Theorem 5.8 by Proposition 1.1. □

Remark. Again, this implies the pfaffian cases $t^{1/2} \in \{\pm p^{1/2}, \pm q^{1/2}\}$.

The corresponding "vanishing" result states that the integral

$$\langle \tilde{\mathcal{R}}^{(n)}_{\lambda}(\dots, p^{1/4}z_i^{\pm 1}, \dots; p^{1/4}t_0{:}p^{-1/4}t_0, p^{1/4}t_1, p^{-1/4}t_1; p^{1/4}u_0, p^{-1/4}u_0; t; p, q)\rangle^{(n/2)}_{t_0,t_1,u_0,1,t^{1/2},q^{1/2}t^{1/2};t;p^{1/2},q} \tag{5.76}$$

or

$$\langle \tilde{\mathcal{R}}^{(n)}_{\lambda}(\dots, p^{1/4}z_i^{\pm 1}, \dots, p^{1/4}; p^{1/4}t_0{:}p^{-1/4}t_0, p^{1/4}t_1, p^{-1/4}t_1; p^{1/4}u_0, p^{-1/4}u_0; t; p, q)\rangle^{((n-1)/2)}_{t_0,t_1,u_0,t,t^{1/2},q^{1/2}t^{1/2};t;p^{1/2},q}, \tag{5.77}$$

as appropriate, evaluates to

$$\frac{\Delta_{\boldsymbol{\lambda}}(-1/u_0|t^{n/2}, t^{(n-1)/2}t_0, \pm(t^{n-1}t_0u_0)^{-1/2}; t^{1/2}; p^{1/2}, q^{1/2})}{\Delta_{\boldsymbol{\lambda}}(1/u_0^2|t^n, t^{n-1}t_0^2, 1/t^{n-1}t_0u_0, 1/t^{n-1}t_0u_0; t; p, q)}. \tag{5.78}$$

Again, for $n = 1$, this is a known quadratic evaluation [28, (1.4)]. If one sets $t_0 = \sqrt{q}$, the biorthogonal function becomes an interpolation polynomial; taking $t_1, u_0 \sim p^{1/4}$ and $p \to 0$ gives a conjecture for Macdonald polynomials which for n even reads

$$\frac{1}{Z}\int P_\lambda(\dots, z_i^{\pm 1}, \dots; q, t) \prod_{1\le i<j\le n/2} \frac{(z_i^{\pm 1}z_j^{\pm 1}; q)}{(tz_i^{\pm 1}z_j^{\pm 1}; q)} \prod_{1\le i\le n/2} \frac{(-z_i^{\pm 1}; q^{1/2})}{(t^{1/2}z_i^{\pm 1}; q^{1/2})} \frac{dz_i}{2\pi\sqrt{-1}z_i}$$
$$= \frac{C^0_\lambda(t^{n/2}; q^{1/2}, t^{1/2}) C^-_\lambda(-q^{1/2}; q^{1/2}, t^{1/2})}{C^0_\lambda(-q^{1/2}t^{(n-1)/2}; q^{1/2}, t^{1/2}) C^-_\lambda(t^{1/2}; q^{1/2}, t^{1/2})}. \tag{5.79}$$

(For a proof in the special case $q = 0$, see [25, Cor. 6.4].) If we replace the integrand by the right-hand side of the Cauchy identity, then take the limit $n \to \infty$, this again becomes Kawanaka's conjecture. The case $t^{1/2} = -q^{1/2}$ of the Macdonald polynomial conjecture is also of interest, as it gives the well-known identity

$$\int_{O\in O(n)} \det(1 + O) s_\lambda(O) = 1. \tag{5.80}$$

The conjecture obtained from Conjecture Q5 in the corresponding way is the same, except with p and q swapped. (The fact that this changes a sum of two integrals to a single integral should not be a concern, since after all Conjectures Q4 and Q5 are each other's modular transforms.) We thus have only one more transform to consider, namely that obtained from Conjecture Q3.

Conjecture Q7 $(t^{-1/2},-1)$**.** *Subject to the balancing conditions $t^{n-1}t_0t_1t_2u_0 = pqt^{1/2}$, $v_0v_1 = p^{1/2}q^{1/2}/t^{1/2}$, one has*

$$\begin{aligned}
&\int \mathcal{R}^{*(n)}_{\boldsymbol\lambda}(;t^{-1/4}t_0,t^{-1/4}u_0;t;p,q)\\
&\quad\times\Delta^{(n)}(;t^{-1/4}t_0,t^{-1/4}t_1,t^{-1/4}t_2,t^{-1/4}u_0,t^{1/4},p^{1/2}t^{1/4},q^{1/2}t^{1/4},p^{1/2}q^{1/2}t^{1/4},-t^{1/4}v_0^2,-t^{1/4}v_1^2;t^{1/2};p,q)\\
&=\frac{\Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0/u_0|t^{n-1}t^{-1/2}t_0t_1;t;p,q)}{\Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0/u_0|t^{n-1}t_0t_1;t;p,q)}\prod_{0\le i<n}\prod_{0\le r<s<3}\frac{\Gamma_{p,q}(t^{i-1/2}t_rt_s)}{\Gamma_{p,q}(t^it_rt_s)}\\
&\quad\times\int \mathcal{R}^{*(n)}_{\boldsymbol\lambda}(\dots,z_i^2,\dots;-t_0,-u_0;t;p,q)\Delta^{(n)}(;\pm\sqrt{-t_0},\pm\sqrt{-t_1},\pm\sqrt{-t_2},\pm\sqrt{-u_0},v_0,v_1;t^{1/2};p^{1/2},q^{1/2}).
\end{aligned}\tag{5.81}$$

Proposition 5.12. *Conjecture Q7 holds for $n=1$.*

Proof. Exchange order of integration in the double integral

$$\int\int \Gamma_{p,q}(t^{1/4}y^{\pm 2}x^{\pm 1})\frac{\prod_{0\le r<4}\Gamma_{p,q}(u_rx^{\pm 1})}{\Gamma_{p,q}(x^{\pm 2})}\frac{dx}{2\pi\sqrt{-1}x}\frac{\Gamma_{p^{1/2},q^{!/2}}(v_0y^{\pm 1},v_1y^{\pm 1})}{\Gamma_{p^{1/2},q^{!/2}}(y^{\pm 2})}\frac{dy}{2\pi\sqrt{-1}y},\tag{5.82}$$

then use gamma function identities to express the integrands in the standard form. ☐

Remark. Again, Van de Bult [4] has recently proved the case $\boldsymbol\lambda=0$ of Conjecture Q7, which by the usual considerations implies the general $n\le 3$ case and the pfaffian cases $t^{1/2}\in\{\pm p^{1/2},\pm q^{1/2}\}$.

The corresponding "vanishing" integral states that

$$\langle \tilde R^{(n)}_{\boldsymbol\lambda}(;t^{-1/4}t_0{:}t^{1/4}t_0,t^{-1/4}t_1,t^{1/4}t_1;t^{-1/4}u_0,t^{1/4}u_0;t;p,q)\rangle^{(n)}_{t^{-1/4}t_0,t^{-1/4}t_1,t^{-1/4}u_0,t^{1/4},p^{1/2}t^{1/4},q^{1/2}t^{1/4};t^{1/2};p,q}\tag{5.83}$$

takes value (5.78).

We close with a combinatorial remark. The "Q" conjectures, if we count both forms of those integrals not symmetric between p and q, give rise to twelve conjectures, one for each ordered pair (a,b) with $a\ne b\in\{-1,p^{1/2},q^{1/2},t^{-1/2}\}$. Furthermore, the three involutions "modular transform", "swap p and q", and "dualize" act on the labels via their natural action on this set of square roots. In addition, in the conjecture associated to the pair (a,b), the integrals are related by a factor

$$\frac{\Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0/u_0|t^{n-1}t_0t_1a;t;p,q)}{\Delta^0_{\boldsymbol\lambda}(t^{n-1}t_0/u_0|t^{n-1}t_0t_1/b^2;t;p,q)}\prod_{\substack{0\le i<n\\0\le r<s<3}}\frac{\Gamma_{p,q}(t^it_rt_ss)}{\Gamma_{p,q}(t^it_rt_s/b^2)},\tag{5.84}$$

with balancing condition $t^{n-1}t_0t_1t_2u_0=pqb^2/a$. This pattern, together with corresponding patterns in the parameters of the interpolation functions, allows us to verify consistency with respect to the Pieri identity and the connection coefficient identity for all of the cases at once, apart from checking that multiplying the interpolation functions by

$$\prod_{1\le i\le n}\frac{\Gamma_{p,q}(a^{1/2}Qt_0z_i^{\pm 1},a^{1/2}u_0z_i^{\pm 1}/Q)}{\Gamma_{p,q}(a^{1/2}t_0z_i^{\pm 1},a^{1/2}u_0z_i^{\pm 1})},\quad\text{or}\quad\prod_{1\le i\le n}\frac{\Gamma_{p,q}(Qt_0z_i^{\pm 1}/b,u_0z_i^{\pm 1}/bQ)}{\Gamma_{p,q}(t_0z_i^{\pm 1}/b,u_0z_i^{\pm 1}/b)},\tag{5.85}$$

as appropriate, before specializing, has the effect, after specializing, of shifting parameters in the corresponding integrand. Similarly, the L conjectures correspond to four identities in natural bijection with the above four square roots. Unfortunately, there are enough quirks in the various cases to make it unclear how to formulate the conjectures in a more uniform manner.

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